Mathematics 2423-001H Name (please print) ____________________

Examination III

April 28, 2005

Instructions: The exam might be on the long side, so avoid spending a lot of time on any individual problem unless you have completed all the other problems that you definitely know how to do. That is, grab easy points first.

I. (10) Let $\sin^{-1}(x)$ be the inverse of the function $f(x) = \sin(x)$, $-\pi/2 \le x \le \pi/2$.

1. Find the domain and range of $\sin^{-1}(x)$.
2. Sketch the graph of $\sin^{-1}(x)$.
3. Use right triangles to simplify the expressions $\cos(\sin^{-1}(x))$ and $\cot\left(\sin^{-1}\left(\frac{\sqrt{x+2}}{x}\right)\right)$.
4. Use the chain rule to calculate the derivative of $\sin^{-1}(x)$, and write the corresponding indefinite integral formula.

II. (10) On one x-y coordinate system, sketch the graphs of $\sinh(x)$ and $\cosh(x)$. Explain why $(\cosh(t), \sinh(t))$ is a point on a hyperbola, and on a second x-y coordinate system sketch that hyperbola and a typical point of the form $(\cosh(t), \sinh(t))$, indicating what t equals geometrically.

III. (12) Use l'Hôpital's rule to calculate the following limits:

1. $\lim_{x\to 0} \frac{\tan(px)}{\tan(qx)}$.
2. $\lim_{x\to 0} \sin(x)\ln(x)$.
3. $\lim_{x\to 0} x^{\sqrt{x}}$.

IV. (15) Calculate the following integrals.

1. $\int x^2 \sin(x)\, dx$
2. $\int \sin^3(mx)\, dx$
3. $\int \sin^2(x)\cos^2(x)\, dx$

V. (10) Calculate the following integral by using the substitution $t = \sqrt{2}\tan(\theta)$. Express the answer in terms of t:

$$\int \frac{t^3}{\sqrt{t^2+2}}\, dt.$$

VI. (12) For each of the following rational functions, write out the form of the partial fraction decomposition. Do not solve for unknown values of the coefficients.

1. $\frac{x^3}{x^4-1}$
2. $\frac{1}{x^3+2x^2+x}$

3. $\dfrac{x^2}{x^3+1}$

VII. (6) Evaluate $\displaystyle\int \frac{\sec^2(\theta)\tan^2(\theta)}{\sqrt{9-\tan^2(\theta)}}\,d\theta$ by using one of the following formulas from the table of integrals:

1. $\displaystyle\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

2. $\displaystyle\int \frac{u^2}{\sqrt{2au-u^2}}\,du = -\frac{u+3a}{2}\sqrt{2au-u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

3. $\displaystyle\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$

4. $\displaystyle\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

VIII. (12) Suppose that $f(x)$ is a function whose third derivative $f^{(3)}(x)$ exists and is continuous. Define $E_2(h)$ by the formula $f(a+h) = f(a) + f'(a)h + \frac{1}{2!}f''(a)h^2 + E_2(h)$.

1. Use integration by parts to calculate that $E_2(h) = \displaystyle\int_0^h \frac{1}{2!}(h-t)^2\, f^{(3)}(a+t)\,dt$.

2. Let m be the minimum and M the maximum of $f^{(3)}$ on the interval $[a, a+h]$. Show that $\frac{1}{3!}h^3\, m \le E_2(h) \le \frac{1}{3!}h^3\, M$.

3. Use the Intermediate Value Theorem to show that there exists c in $[a, a+h]$ so that $E_2(h) = \frac{1}{3!}f^{(3)}(c)\,h^3$.