Instructions: The exam might be on the long side, so avoid spending a lot of time on any individual problem unless you have completed all the other problems that you definitely know how to do. That is, grab easy points first.

I. Let $\sin^{-1}(x)$ be the inverse of the function $f(x) = \sin(x)$, $-\pi/2 \leq x \leq \pi/2$.

1. Find the domain and range of $\sin^{-1}(x)$.
2. Sketch the graph of $\sin^{-1}(x)$.
3. Use right triangles to simplify the expressions $\cos(\sin^{-1}(x))$ and $\cot\left(\sin^{-1}\left(\frac{\sqrt{x} + 2}{x}\right)\right)$.
4. Use the chain rule to calculate the derivative of $\sin^{-1}(x)$, and write the corresponding indefinite integral formula.

II. On one $x$-$y$ coordinate system, sketch the graphs of $\sinh(x)$ and $\cosh(x)$. Explain why $(\cosh(x), \sinh(x))$ is a point on a hyperbola, and on a second $x$-$y$ coordinate system sketch that hyperbola and a typical point of the form $(\cosh(t), \sinh(t))$, indicating what $t$ equals geometrically.

III. Use l’Hôpital’s rule to calculate the following limits:

1. $\lim_{x \to 0} \frac{\tan(px)}{\tan(qx)}$.
2. $\lim_{x \to 0} \sin(x) \ln(x)$.
3. $\lim_{x \to 0} x^{\sqrt{x}}$.

IV. Calculate the following integrals.

1. $\int x^2 \sin(x) \, dx$
2. $\int \sin^3(mx) \, dx$
3. $\int \sin^2(x) \cos^2(x) \, dx$

V. Calculate the following integral by using the substitution $t = \sqrt{2} \tan(\theta)$. Express the answer in terms of $t$:

$\int \frac{t^3}{\sqrt{t^2 + 2}} \, dt$.

VI. For each of the following rational functions, write out the form of the partial fraction decomposition. Do not solve for unknown values of the coefficients.

1. $\frac{x^3}{x^4 - 1}$
2. $\frac{1}{x^3 + 2x^2 + x}$
3. \( \frac{x^2}{x^3 + 1} \)

VII. Evaluate \( \int \frac{\sec^2(\theta) \tan^2(\theta)}{\sqrt{9 - \tan^2(\theta)}} \, d\theta \) by using one of the following formulas from the table of integrals:

(6)

1. \( \int \frac{u^2}{\sqrt{u^2 - a^2}} \, du = \frac{u}{2} \sqrt{u^2 - a^2} + \frac{a^2}{2} \ln |u + \sqrt{u^2 - a^2}| + C \)

2. \( \int \frac{u^2}{\sqrt{2au - u^2}} \, du = -\frac{u + 3a}{2} \sqrt{2au - u^2} + \frac{3a^2}{2} \cos^{-1} \left( \frac{a - u}{a} \right) + C \)

3. \( \int \frac{u^2}{\sqrt{a^2 - u^2}} \, du = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{u}{a} \right) + C \)

4. \( \int \frac{u^2}{\sqrt{a + bu}} \, du = \frac{2}{15b^3} (8a^2 + 3b^2u^2 - 4abu) \sqrt{a + bu} + C \)

VIII. Suppose that \( f(x) \) is a function whose third derivative \( f^{(3)}(x) \) exists and is continuous. Define \( E_2(h) \) by the formula \( f(a + h) = f(a) + f'(a)h + \frac{1}{2!} f''(a)h^2 + E_2(h) \).

1. Use integration by parts to calculate that \( E_2(h) = \int_0^h \frac{1}{2!} (h - t)^2 f^{(3)}(a + t) \, dt \).

2. Let \( m \) be the minimum and \( M \) the maximum of \( f^{(3)} \) on the interval \( [a, a + h] \). Show that \( \frac{1}{3!} h^3 m \leq E_2(h) \leq \frac{1}{3!} h^3 M \).

3. Use the Intermediate Value Theorem to show that there exists \( c \) in \( [a, a + h] \) so that \( E_2(h) = \frac{1}{3!} f^{(3)}(c) h^3 \).