Mathematics 2423-001H Name (please print) ______________________

Examination II

March 24, 2005

I. (6) Let R be the region bounded by $y = \dfrac{1}{x^2+1}$, the x-axis, the y-axis, and the line $x = 2$. Calculate the volume produced when this region is rotated about the y-axis.

Using "cylindrical shells", that is, cross-sections obtained when vertical line segments are rotated about the y-axis, the cross-sectional area is $A(x) = 2\pi x \cdot \dfrac{1}{x^2+1}$, so the volume is

$$\int_0^2 \pi \cdot \frac{2x}{x^2+1}\, dx = \pi \ln(x^2+1)\Big|_0^2 = \pi(\ln(5) - \ln(1)) = \pi \ln(5)\ .$$

II. (6) Calculate the volume produced when the region bounded by the curve $y = e^x$, the x-axis, the y-axis, and the line $x = 1$ is rotated about the line $y = -c$, where c is a positive number.

Cross-sections obtained when vertical line segments are rotated about $y = -c$ are annuli with outer radius $c + e^x$ and inner radius c, so the cross-sectional area is $A(x) = \pi(c+e^x)^2 - \pi c^2 = 2\pi c e^x + \pi e^{2x}$. The resulting volume is

$$\int_0^1 2\pi c e^x + \pi e^{2x}\, dx = 2\pi c e^x + \frac{\pi}{2} e^{2x}\Big|_0^1 = \frac{\pi}{2}(e^2 - 1) + 2\pi c(e-1)\ .$$

III. (6) Define what it means to say that a function f is *one-to-one*. Find the smallest value of a for which the function defined by $f(x) = 3x^2 + 17x + 217$ is one-to-one on the interval $[a, \infty)$ (i. e. for $a \le x < \infty$).

f is *one-to-one* when $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. The graph of $f(x) = 3x^2 + 17x + 217$ is a parabola that opens upward, so a would be the point where the minimum value occurs. This is the point where $f'(x) = 6x + 17 = 0$, that is, $a = -\frac{17}{6}$.

IV. (7) Let $a > 1$. Explain why the derivative of the function a^x at $x = 0$ is $\displaystyle\lim_{h\to 0} \frac{a^h - 1}{h}$. Writing a_0 for the number $\displaystyle\lim_{h\to 0} \frac{a^h - 1}{h}$, show that the derivative of a^x is $a_0\, a^x$.

Calculating the derivative as a limit of difference quotients gives the derivative of a^x at $x = 0$ to be

$$\lim_{h\to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h\to 0} \frac{a^h - 1}{h}\ .$$

At an arbitrary specific value of x, the derivative is then

$$\lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h} = \lim_{h\to 0} a^x \frac{a^h - 1}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a_0 a^x\ .$$

V. Solve the following equations for x.
(6)

1. $2\ln(x) = \ln(2) + \ln(x+1)$.

   Using properties of logarithm, the equation becomes $\ln(x^2) = \ln(2(x+1))$. Exponentiating both sides gives $x^2 = 2x+2$ so $x^2 - 2x - 2 = 0$. The quadratic formula gives $x = \dfrac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$. Since $x > 0$ in order to appear in $\ln(x)$, we must have $x = 1 + \sqrt{3}$.

2. $e^{ax} = Ce^{bx}$.

   Taking logarithms of both sides gives $ax = \ln(C) + bx$, so $ax - bx = \ln(C)$ and $x = \dfrac{\ln(C)}{a-b}$.

VI. Calculate the following derivatives.
(12)

1. $\dfrac{dy}{dt}$ if $y = \ln\left(\sqrt[4]{\left(\dfrac{(2t+1)^3}{t^2-1}\right)^5}\right)$.

   Using properties of logarithm, we have

   $$y = \ln\left(\left(\frac{(2t+1)^3}{t^2-1}\right)^{5/4}\right) = \frac{5}{4}(\ln((2t+1)^3) - \ln(t^2-1)) = \frac{5}{4}(3\ln(2t+1) - \ln(t^2-1)) \ ,$$

   so

   $$\frac{dy}{dt} == \frac{5}{4}\left(3\frac{1}{2t+1}\cdot 2 - \frac{1}{t^2-1}\cdot 2t\right) = \frac{5}{2}\left(\frac{3}{2t+1} - \frac{t}{t^2-1}\right) \ .$$

2. $\dfrac{dy}{dx}$ if $y = x^{1/x}$.

   Applying the logarithm to both sides of $y = x^{1/x}$ gives $\ln(y) = \ln(x^{1/x}) = \dfrac{1}{x}\ln(x)$. Differentiating implicitly gives

   $$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}\ln(x) + \frac{1}{x}\cdot\frac{1}{x} = \frac{1-\ln(x)}{x^2} \ ,$$

   so $\frac{dy}{dx} = y\,\dfrac{1-\ln(x)}{x^2} = x^{1/x}\,\dfrac{1-\ln(x)}{x^2}$.

3. $\dfrac{d}{dx}\big(f^{-1}(x)\big)$ in terms of f'.

   Differentiating the fundamental relationship $f(f^{-1}(x)) = x$, using the Chain Rule, we obtain

   $$f'(f^{-1}(x)) \cdot \frac{d}{dx}\big(f^{-1}(x)\big) = 1 \ .$$

   Therefore $\dfrac{d}{dx}\big(f^{-1}(x)\big) = \dfrac{1}{f'(f^{-1}(x))}$, which cannot be simplified without further information.

VII. (12) Calculate the following integrals.

1. $\displaystyle\int \frac{e^x+1}{e^x}\,dx.$

$$\int \frac{e^x+1}{e^x}\,dx = \int 1+e^{-x}\,dx = x - e^{-x} + C.$$

2. $\displaystyle\int \frac{e^x}{e^x+1}\,dx.$

Using $u = e^x+1$ and $du = e^x\,dx$, we have $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{1}{u}\,du = \ln(|u|) + C = \ln(e^x+1) + C.$

3. $\displaystyle\int x\,2^{x^2}\,dx.$

Using $u = x^2$ and $du = 2x\,dx$, we have $\displaystyle\int x\,2^{x^2}\,dx = \frac{1}{2}\int 2^u\,du = \frac{1}{2\ln(2)}2^u + C = \frac{1}{2\ln(2)}2^{x^2} + C.$

4. $\displaystyle\int_e^6 \frac{dx}{x\,\ln(x)}.$

Using $u = \ln(x)$ and $du = \frac{1}{x}\,dx$, we have $\displaystyle\int_e^6 \frac{dx}{x\,\ln(x)} = \int_1^{\ln(6)} \frac{1}{u}\,du = \ln(u)\Big|_1^{\ln(6)} = \ln(\ln(6)).$

VIII. (5) For $x > 1$, let $M(x)$ be the average value of the natural logarithm function on the interval from 1 to x. Write an expression for $M(x)$. Verify that $M(x) + (x-1)M'(x) = \ln(x)$.

$M(x) = \displaystyle\frac{1}{x-1}\int_1^x \ln(t)\,dt$. Writing this as $(x-1)M(x) = \displaystyle\int_1^x \ln(t)\,dt$ and differentiating using the product rule and the Fundamental Theorem of Calculus gives $M(x) + (x-1)M'(x) = \ln(x)$.

IX. (5) How are the volume of an object and its average cross-sectional area related?

Let $A(x)$ be the cross-sectional area, where the range $a \le x \le b$ describes the object. The average cross-sectional area is then $\frac{1}{b-a}\int_a^b A(x)\,dx$, and the volume is

$$\int_a^b A(x)\,dx = (b-a)\cdot\frac{1}{b-a}\int_a^b A(x)\,dx\ .$$

That is, the volume is the length times the average cross-sectional area.