I. Use the telescoping sum \( \sum_{k=1}^{n} k^2 - (k-1)^2 \) to obtain the formula \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \).

\[
n^2 = \sum_{k=1}^{n} k^2 - (k-1)^2 = \sum_{k=1}^{n} 2k - 1 = 2 \sum_{k=1}^{n} k - \sum_{k=1}^{n-1} k = 2 \sum_{k=1}^{n} k - n. \]

Solving for \( \sum_{k=1}^{n} k \), we find that \( \sum_{k=1}^{n} k = \frac{1}{2} (n^2 + n) = \frac{n(n+1)}{2} \).

II. Calculate the Riemann sum for the following partition and function, using left-hand endpoints as the sample points \( x_i^* \): the function is \( f(x) = x^2 / 2 \), the interval is \([1, 10]\), and the partition is \( x_1 = 2, x_2 = 4, \) and \( x_3 = 9 \).

\[
\sum_{i=1}^{4} f(x_i^*) \Delta x_i = f(1) \cdot 1 + f(2) \cdot 2 + f(4) \cdot 5 + f(9) \cdot 1 = \frac{1}{2} + 4 + 40 + \frac{81}{2} = 85.
\]

III. Give an explicit example of a partition of the interval \([0, 10]\) that has mesh \( \pi \).

\( \pi, 4, 5, 6, 7, 8, 9, \) or \( \pi, 6, 8, \) etc.

IV. Let \( f(x) \) be the function defined by \( f(x) = 0 \) for \( 0 \leq x < 5 \) and \( f(x) = 1 \) for \( 5 \leq x \leq 10 \). Consider the partition of \([0, 10]\) defined by \( x_1 = 3, x_2 = 7 \). By making two different choices of the points \( x_i^* \), show that both of the numbers 3 and 7 are Riemann sums for this function and this partition of \([0, 10]\).

For the first partition, choose (for example) \( x_1^* = 0, x_2^* = 3, \) and \( x_3^* = 10 \), giving the Riemann sum \( f(x_1^*) \cdot \Delta x_1 + f(x_2^*) \cdot \Delta x_2 + f(x_3^*) \cdot \Delta x_3 = 0 + 0 + 1 \cdot 3 = 3 \). For the second, we could take \( x_1^* = 0, x_2^* = 7, \) and \( x_3^* = 10 \), giving the Riemann sum \( f(x_1^*) \cdot \Delta x_1 + f(x_2^*) \cdot \Delta x_2 + f(x_3^*) \cdot \Delta x_3 = 0 + 1 \cdot 4 + 1 \cdot 3 = 7 \).

V. Write the following limit as an integral, but do not try to calculate the integral: \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{6n} \tan \left( \frac{i\pi}{6n} \right) \).

These are Riemann sums for (among other possibilities) the equal-length partition of \([a, b] = [0, \pi/6]\), with \( \Delta x_i = \frac{\pi/6}{n} = \frac{\pi}{6n} \), \( f(x) = \tan(x) \) and \( x_i^* = i \cdot \frac{\pi}{6n} \). So the limit is \( \int_{0}^{\pi/6} \tan(x) \, dx \).

VI. State the Fundamental Theorem of Calculus (both parts, of course).

Let \( f \) be a continuous function on \([a, b]\). Then \( \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x) \). If \( F' = f \) on \([a, b]\), then \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \).

VII. State the Mean Value Theorem for Integrals.

Let \( f \) be a continuous function on \([a, b]\). Then there exists a number \( c \) between \( a \) and \( b \) for which \( \int_{a}^{b} f(x) \, dx = f(c)(b - a) \).
The problem can also be solved using the Fundamental Theorem of Calculus: By the FTC(b), we have 

\[
\frac{d}{dx} \int_1^x \frac{\sin(t)}{t} \, dt = \frac{\sin(x)}{x}.
\]

For the second, the Chain Rule shows that the derivative of the first is \(\frac{\sin(x^3)}{x}\). For the last one, we write the integral as 

\[
\int_1^x \frac{\sin(t)}{t} \, dt + \int_1^{t^3} \frac{\sin(t)}{t} \, dt = \int_1^{t^3} \frac{\sin(t)}{t} \, dt - \int_1^t \frac{\sin(t)}{t} \, dt,
\]

and using the Chain Rule as in the second case we find the derivative to be 

\[
\frac{3 \sin(x^3)}{x} - \frac{2 \sin(x^2)}{x}.
\]

Verify that \(\int (x^2 - 1)^{3/2} \, dx\) is not \(\frac{2}{5}(x^2 - 1)^{5/2} + C\).

The derivative of \(\frac{2}{5}(x^2 - 1)^{5/2} + C\) is 

\[(x^2 - 1)^{3/2} \cdot 2x,\]

and since this does not equal \((x^2 - 1)^{3/2}\) (for example, they have different values at \(x = 2\)), 

\(\int (x^2 - 1)^{3/2} \, dx \neq \frac{2}{5}(x^2 - 1)^{5/2} + C\).

Calculate the following indefinite integrals: 

\[
\int (w + \frac{1}{w})^2 \, dw, \quad \int \sqrt{\cot(x)} \csc^2(x) \, dx, \quad \text{and} \quad \int \frac{\cos(\pi/x)}{x^2} \, dx.
\]

\[
\int (w + \frac{1}{w})^2 \, dw = \int w^2 + 2 + w^{-2} \, dw = \frac{w^3}{3} + 2w - \frac{1}{w} + C.
\]

Using the substitution \(u = \cot(x)\) and \(du = -\csc^2(x) \, dx\), we have 

\[
\int \sqrt{\cot(x)} \csc^2(x) \, dx = \int -u^{1/2} \, du = \frac{u^{3/2}}{3} + C = -\frac{2\cot^{3/2}(x)}{3} + C.
\]

Using the substitution \(u = \frac{\pi}{x}\) and \(du = -\frac{\pi}{x^2} \, dx\), we have 

\[
\int \frac{\cos(\pi/x)}{x^2} \, dx = \int -\frac{1}{\pi} \cos(u) \, du = -\frac{1}{\pi} \sin(u) + C = -\frac{1}{\pi} \sin(\pi/x) + C.
\]

Calculate \(\int_0^{3\pi/2} |\cos(\theta)| \, d\theta\).

\[
\int_0^{3\pi/2} |\cos(\theta)| \, d\theta = \int_0^{\pi} |\cos(\theta)| \, d\theta + \int_{\pi}^{3\pi/2} |\cos(\theta)| \, d\theta = \int_0^{\pi} \cos(\theta) \, d\theta + \int_{\pi/2}^{3\pi/2} -\cos(\theta) \, d\theta = \sin(\pi/2) - \sin(0) + (\sin(3\pi/2) - (-\sin(\pi/2))) = (1 - 0) + (-1) - (-1) = 3.
\]

A differentiable function \(f(x)\) satisfies \(f(100) = 100\) and \(f'(x) < \frac{1}{x}\) for all \(x\). Show that \(f(1000) < 109\).

By the Mean Value Theorem, we have 

\[
f(1000) - f(100) = f'(c) \cdot (1000 - 100) = 900f'(c) \quad \text{for some} \quad c \quad \text{between} \quad 100 \quad \text{and} \quad 1000.
\]

Since \(100 < c\), we have 

\[
f'(c) < \frac{1}{c} < \frac{1}{100}, \quad \text{so} \quad 900f'(c) < 900 \cdot \frac{1}{100} = 9.
\]

So 

\[
f(1000) = f(100) + 900f'(c) < 109.
\]

The problem can also be solved using the Fundamental Theorem of Calculus: By the FTC(b), we have 

\[
f(1000) - f(100) = \int_{100}^{1000} f'(x) \, dx \leq \int_{100}^{1000} \frac{1}{x} \, dx < \int_{100}^{1000} \frac{1}{100} \, dx = 900 \cdot \frac{1}{100} = 9, \quad \text{so} \quad f(1000) < f(100) + 9 = 109.
\]
XIII. Use the substitution $u = \sin(\theta)$ (and the fact that $\cos^2(\theta) = 1 - \sin^2(\theta)$) to calculate that the following integral $\int_{0}^{\pi} \sin^5(\theta) \cos^7(\theta) \, d\theta$ equals 0.

When $u = \sin(\theta)$, we have $du = \cos(\theta) \, d\theta$ and $\cos^2(\theta) = 1 - \sin^2(\theta) = 1 - u^2$, so $\int_{0}^{\pi} \sin^5(\theta) \cos^7(\theta) \, d\theta = \int_{0}^{\pi} \sin^5(\theta) \cos^6(\theta) \cos(\theta) \, d\theta = \int_{0}^{0} u^5 (1 - u^2)^3 \, du = 0.$

XIV. Simplify $x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots + x^{202}$.

$(x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots + x^{202})(1 + x^2) = x^2 + x^{204}$, so $x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots + x^{202} = \frac{x^2 + x^{204}}{1 + x^2}.$