Mathematics 2423-001H Name (please print) ____________________

Examination I

February 15, 2005

I. (5) Use the telescoping sum $\displaystyle\sum_{k=1}^{n} k^2 - (k-1)^2$ to obtain the formula $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

$n^2 = \displaystyle\sum_{k=1}^{n} k^2 - (k-1)^2 = \sum_{k=1}^{n} 2k - 1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\sum_{k=1}^{n} k - n$. Solving for $\displaystyle\sum_{k=1}^{n} k$, we find that $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}(n^2+n) = \frac{n(n+1)}{2}$.

II. (5) Calculate the Riemann sum for the following partition and function, using left-hand endpoints as the sample points x_i^*: the function is $f(x) = x^2/2$, the interval is $[1,10]$, and the partition is $x_1 = 2$, $x_2 = 4$, and $x_3 = 9$.

$$\sum_{i=1}^{4} f(x_i^*)\,\Delta x_i = f(1)\cdot 1 + f(2)\cdot 2 + f(4)\cdot 5 + f(9)\cdot 1 = \frac{1}{2} + 4 + 40 + \frac{81}{2} = 85.$$

III. (3) Give an explicit example of a partition of the interval $[0,10]$ that has mesh π.

$\pi, 4, 5, 6, 7, 8, 9$, or $\pi, 6, 8$, etc.

IV. (5) Let $f(x)$ be the function defined by $f(x) = 0$ for $0 \le x < 5$ and $f(x) = 1$ for $5 \le x \le 10$. Consider the partition of $[0,10]$ defined by $x_1 = 3$, $x_2 = 7$. By making two different choices of the points x_i^*, show that both of the numbers 3 and 7 are Riemann sums for this function and this partition of $[0,10]$.

For the first partition, choose (for example) $x_1^* = 0$, $x_2^* = 3$, and $x_3^* = 10$, giving the Riemann sum $f(x_1^*)\cdot\Delta x_1 + f(x_2^*)\cdot\Delta x_2 + f(x_3^*)\cdot\Delta x_3 = 0 + 0 + 1\cdot 3 = 3$. For the second, we could take $x_1^* = 0$, $x_2^* = 7$, and $x_3^* = 10$, giving the Riemann sum $f(x_1^*)\cdot\Delta x_1 + f(x_2^*)\cdot\Delta x_2 + f(x_3^*)\cdot\Delta x_3 = 0 + 1\cdot 4 + 1\cdot 3 = 7$.

V. (3) Write the following limit as an integral, but do not try to calculate the integral: $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{6n}\tan\left(\frac{i\pi}{6n}\right)$.

These are Riemann sums for (among other possibilities) the equal-length partition of $[a,b] = [0,\pi/6]$, with $\Delta x_i = \dfrac{\pi/6}{n} = \dfrac{\pi}{6n}$, $f(x) = \tan(x)$ and $x_i^* = i\cdot\dfrac{\pi}{6n}$. So the limit is $\displaystyle\int_0^{\pi/6}\tan(x)\,dx$.

VI. (6) State the Fundamental Theorem of Calculus (both parts, of course).

Let f be a continuous function on $[a,b]$. Then $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. If $F' = f$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

VII. (3) State the Mean Value Theorem for Integrals.

Let f be a continuous function on $[a,b]$. Then there exists a number c between a and b for which $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a)$.

VIII. Calculate the following derivatives: $\dfrac{d}{dx}\displaystyle\int_1^x \frac{\sin(t)}{t}\,dt$, $\dfrac{d}{dx}\displaystyle\int_1^{t^3} \frac{\sin(t)}{t}\,dt$, $\dfrac{d}{dx}\displaystyle\int_{t^2}^{t^3} \frac{\sin(t)}{t}\,dt$.
(6)

Direct application of the FTC(a) shows that the derivative of the first is $\dfrac{\sin(x)}{x}$. For the second, the Chain Rule shows that the derivative is $\dfrac{\sin(x^3)}{x^3} \cdot \dfrac{d}{dx}(x^3) = \dfrac{3x^2\sin(x^3)}{x^3} = \dfrac{3\sin(x^3)}{x}$. For the last one, we write the integral as $\displaystyle\int_{t^2}^1 \frac{\sin(t)}{t}\,dt + \int_1^{t^3} \frac{\sin(t)}{t}\,dt = \int_1^{t^3} \frac{\sin(t)}{t}\,dt - \int_1^{t^2} \frac{\sin(t)}{t}\,dt$, and using the Chain Rule as in the second case we find the derivative to be $\dfrac{3\sin(x^3)}{x} - \dfrac{2\sin(x^2)}{x}$.

IX. Verify that $\displaystyle\int (x^2-1)^{3/2}\,dx$ is **not** $\frac{2}{5}(x^2-1)^{5/2} + C$.
(3)

The derivative of $\frac{2}{5}(x^2-1)^{5/2} + C$ is $(x^2-1)^{3/2} \cdot 2x$, and since this does not equal $(x^2-1)^{3/2}$ (for example, they have different values at $x = 2$), $\displaystyle\int (x^2-1)^{3/2}\,dx \neq \frac{2}{5}(x^2-1)^{5/2} + C$.

X. Calculate the following indefinite integrals: $\displaystyle\int \left(w + \frac{1}{w}\right)^2 dw$, $\displaystyle\int \sqrt{\cot(x)}\csc^2(x)\,dx$, and $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$.
(9)

$$\int \left(w + \frac{1}{w}\right)^2 dw = \int w^2 + 2 + w^{-2}\,dw = \frac{w^3}{3} + 2w - \frac{1}{w} + C.$$

Using the substitution $u = \cot(x)$ and $du = -\csc^2(x)\,dx$, we have $\displaystyle\int \sqrt{\cot(x)}\csc^2(x)\,dx = \int -u^{1/2}\,du = -\frac{u^{3/2}}{3/2} + C = -\frac{2\cot^{3/2}(x)}{3} + C$.

Using the substitution $u = \dfrac{\pi}{x}$ and $du = -\dfrac{\pi}{x^2}\,dx$, we have $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx = \int -\frac{1}{\pi}\cos(u)\,du = -\frac{1}{\pi}\sin(u) + C = -\frac{1}{\pi}\sin(\pi/x) + C$.

XI. Calculate $\displaystyle\int_0^{3\pi/2} |\cos(\theta)|\,d\theta$.
(4)

$\displaystyle\int_0^{3\pi/2} |\cos(\theta)|\,d\theta = \int_0^{\pi} |\cos(\theta)|\,d\theta + \int_{\pi}^{3\pi/2} |\cos(\theta)|\,d\theta = \int_0^{\pi/2} \cos(\theta)\,d\theta + \int_{\pi/2}^{3\pi/2} -\cos(\theta)\,d\theta = \sin(\pi/2) - \sin(0) + (-\sin(3\pi/2) - (-\sin(\pi/2))) = (1-0) + (-(-1) - (-1)) = 3$.

XII. A differentiable function $f(x)$ satisfies $f(100) = 100$ and $f'(x) < \dfrac{1}{x}$ for all x. Show that $f(1000) < 109$.
(5)

By the Mean Value Theorem, we have $f(1000) - f(100) = f'(c) \cdot (1000 - 100) = 900f'(c)$ for some c between 100 and 1000. Since $100 < c$, we have $f'(c) < \dfrac{1}{c} < \dfrac{1}{100}$, so $900f'(c) < 900 \cdot \dfrac{1}{100} = 9$. So $f(1000) = f(100) + 900f'(c) < 109$.

The problem can also be solved using the Fundamental Theorem of Calculus: By the FTC(b), we have $f(1000) - f(100) = \displaystyle\int_{100}^{1000} f'(x)\,dx \leq \int_{100}^{1000} \frac{1}{x}\,dx < \int_{100}^{1000} \frac{1}{100}\,dx = 900 \cdot \frac{1}{100} = 9$, so $f(1000) < f(100) + 9 = 109$.

XIII. Use the substitution $u = \sin(\theta)$ (and the fact that $\cos^2(\theta) = 1 - \sin^2(\theta)$) to calculate that the following
(4) integral $\int_0^{\pi} \sin^5(\theta)\cos^7(\theta)\, d\theta$ equals 0.

When $u = \sin(\theta)$, we have $du = \cos(\theta)\, d\theta$ and $\cos^2(\theta) = 1 - \sin^2(\theta) = 1 - u^2$, so $\int_0^{\pi} \sin^5(\theta)\cos^7(\theta)\, d\theta = \int_0^{\pi} \sin^5(\theta)\cos^6(\theta)\ \cos(\theta) d\theta = \int_0^{0} u^5(1-u^2)^3\, du = 0.$

XIV. Simplify $x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots + x^{202}$.
(4)

$(x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots + x^{202})(1 + x^2) = x^2 + x^{204}$, so $x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots + x^{202} = \dfrac{x^2 + x^{204}}{1 + x^2}$.