Mathematics 2423-001H — Name (please print) ____________________

Final Examination

May 11, 2005

Instructions: Remember that even if you cannot do one part of a problem, you may assume that it is true and use it to do later parts of the problem.

I. Calculate the following integrals using integration by parts.
(8)

1. $\int xe^x\,dx$

   Using integration by parts with $u = x$, $du = dx$, $dv = e^x\,dx$, and $v = e^x$, we find that $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$.

2. $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

   Using integration by parts with $u = x^2$, $du = \frac{x}{\sqrt{1+x^2}}\,dx$, $dv = 2x\,dx$, and $v = \sqrt{1+x^2}$, we find that $\int \frac{x^3}{\sqrt{1+x^2}}\,dx = x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2}\,dx = x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2} + C$.

II. Let $\tan^{-1}(x)$ be the inverse of the function $f(x) = \tan(x)$, $-\pi/2 < x < \pi/2$.
(10)

1. Find the domain and range of $\tan^{-1}(x)$.

   Its domain is the range of $f(x)$, that is, all x values. Its range is the domain of $f(x)$, that is, $-\pi/2 < x < \pi/2$.

2. Sketch the graph of $\tan^{-1}(x)$.

3. Use right triangles to simplify the expressions $\csc(\tan^{-1}(x))$ and $\cos\left(2\tan^{-1}\left(\frac{\sqrt{x}}{2}\right)\right)$.

   $\tan^{-1}(x)$ is an angle in a right triangle whose opposite leg is x and adjacent leg is 1. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{1+x^2}$, giving $\csc(\tan^{-1}(x)) = \sqrt{1+x^2}/x$.
   $\cos\left(2\tan^{-1}\left(\frac{\sqrt{x}}{2}\right)\right) = \cos^2\left(\tan^{-1}\left(\frac{\sqrt{x}}{2}\right)\right) - \sin^2\left(\tan^{-1}\left(\frac{\sqrt{x}}{2}\right)\right)$. To find these, we use an angle in a right triangle whose opposite leg is $\sqrt{x}$ and adjacent leg is 2. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{x+4}$, giving $\cos\left(2\tan^{-1}\left(\frac{\sqrt{x}}{2}\right)\right) = \frac{4}{x+4} - \frac{x}{x+4} = \frac{4-x}{4+x}$.

4. Use the chain rule to calculate the derivative of $\tan^{-1}(x)$, and write the corresponding indefinite integral formula.

   Differentiating the equation $\tan(\tan^{-1}(x)) = x$, we obtain $\sec^2(\tan^{-1}(x)) \cdot \frac{d}{dx}(\tan^{-1}(x)) = 1$, and therefore $\frac{d}{dx}(\tan^{-1}(x)) = \cos^2(\tan^{-1}(x)) = \frac{1}{1+x^2}$ (using a right triangle to find $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$). The corresponding indefinite integral formula is $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$.

III. (8) We know, of course, that the exact value of $\int_0^{\pi} \sin(x)\,dx$ is 2. Calculate the value obtained when Simpson's Rule with $n = 4$ is used to estimate $\int_0^{\pi} \sin(x)\,dx$. (Find the exact value of the estimate; its numerical value is approximately 2.00455.) Use one of the error formulas to estimate the error. (Leave the error estimate as an expression involving π; in case you are curious, its numerical value is close to 0.00664, so it gives a rather accurate estimate of the error.)

The x-values are $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$, $x_3 = 3\pi/4$, and $x_4 = \pi$, with corresponding y-values $y_0 = 0$, $y_1 = 1/\sqrt{2}$, $y_2 = 1$, $y_3 = 1/\sqrt{2}$, and $y_4 = 0$. Plugging into Simpson's rule gives the approximation $\int_0^{\pi} \sin(x)\,dx \approx (\pi/12)(0+4/\sqrt{2}+2+4/\sqrt{2}+0) = \pi(1+2\sqrt{2})/6$. The fourth derivative of $\sin(x)$ is $\sin(x)$, so K is the maximum value of $|\sin(x)|$ for $0 \le x \le \pi$, that is, $K = 1$. We have $b - a = \pi - 0 = \pi$, so the formula from the Table of Integrals gives the maximum possible magnitude of the error to be $\dfrac{\pi^5}{180 \cdot 4^4}$.

IV. (8) Consider the portion of the graph $y = \tan^{-1}(x)$ between $x = 0$ and $x = 1$. For each of the following, write an integral whose value is the specified quantity for this portion of the graph, but do not attempt to evaluate the integrals.

1. The length of this portion of the graph.

   We calculate $ds = \sqrt{1 + \left(\frac{d}{dx}(\tan^{-1}(x))\right)^2}\,dx = \sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}\,dx$. So the length is $\int_0^1 \sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}\,dx$.

2. The surface area obtained when it is rotated about the x-axis.

   $$\int 2\pi\rho\,ds = \int_0^1 2\pi \tan^{-1}(x)\sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}\,dx.$$

3. The surface area obtained when it is rotated about the line $y = -1$.

   $$\int 2\pi\rho\,ds = \int_0^1 2\pi(1 + \tan^{-1}(x))\sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}\,dx.$$

4. The surface area obtained when it is rotated about the y-axis.

   $$\int 2\pi\rho\,ds = \int_0^1 2\pi x\sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}\,dx.$$

V. (8) Use a trig substitution to evaluate the integral $\int \sqrt{1+4x^2}\,dx$. You may want to utilize the table of integrals for some of the later steps in the calculation. Express the answer in terms of x.

Regarding $1+4x^2$ as u^2+a^2 and making the substitution $u = a\tan(\theta)$, we put $2x = \tan(\theta)$, $dx = \frac{1}{2}\sec^2(\theta)\,d\theta$, and compute $\int \sqrt{1+4x^2}\,dx = \int \sqrt{1+\tan^2(\theta)}\frac{1}{2}\sec^2(\theta)\,d\theta = \frac{1}{2}\int \sec^3(\theta)\,d\theta$. Applying formulas from the Table of Integrals, we have $\frac{1}{2}\int \sec^3(\theta)\,d\theta = \frac{1}{2}\cdot\frac{1}{3-1}\tan(\theta)\sec(\theta) + \frac{1}{2}\cdot\frac{3-2}{3-1}\int \sec(\theta)\,d\theta = \frac{1}{4}\tan(\theta)\sec(\theta) + \frac{1}{4}\ln|\sec(\theta)+\tan(\theta)| + C$. Finally, we consider a right triangle with an angle θ whose opposite leg is $2x$ and adjacent leg is 1, so that the hypotenuse is $\sqrt{1+4x^2}$. We observe that $\sec(\theta) = \sqrt{1+4x^2}$. So we have $\frac{1}{4}\tan(\theta)\sec(\theta)+\frac{1}{4}\ln|\sec(\theta)+\tan(\theta)|+C = \frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln|2x+\sqrt{1+4x^2}| + C$.

VI. (9) This problem concerns functions that are one-to-one.

1. Give a formal definition (not just the intuitive idea) of the statement that a function f is *one-to-one*.

   To say that f is one-to-one means that if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

2. Give an example of two functions (defined for all x) that are one-to-one, but whose product is not one-to-one.

   Let $f(x) = g(x) = x$. Each of them is one-to-one, but $f(x)g(x) = x^2$ is not one-to-one.

3. Give an example of two functions (defined for all x) that are not one-to-one, but whose product is one-to-one.

   Let $f(x) = x+1$ for $x \le 0$ and $f(x) = 1$ for $x \ge 0$, and $g(x) = 1$ for $x \le 0$ and $g(x) = x+1$ for $x \ge 0$. Then, $f(x)g(x) = x+1$ for all x, which is one-to-one, but neither $f(x)$ nor $g(x)$ is one-to-one (indeed, each one takes the value 1 at infinitely many points).

VII. (8) Use the table of integrals to calculate $\int \frac{1}{x^2(3x-1)}\,dx$. Calculate $\int_1^\infty \frac{1}{x^2(3x-1)}\,dx$.

Using the formula $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$ with $x = u$, $a = -1$, and $b = 3$, we have $\int \frac{1}{x^2(3x-1)}\,dx = \frac{1}{x} - 3\ln\left|\frac{3x-1}{x}\right| + C = \frac{1}{x} + 3\ln\left|3-\frac{1}{x}\right| + C$. So we have $\int_1^\infty \frac{1}{x^2(3x-1)}\,dx = \lim_{b\to\infty} \frac{1}{x} + 3\ln\left|3-\frac{1}{x}\right|\Big|_1^b = \lim_{b\to\infty}\frac{1}{b} + 3\ln\left|3-\frac{1}{b}\right| - (1+3\ln|3-1|) = 0 + 3\ln(3) - 1 - 3\ln(2) = 3\ln(\frac{3}{2}) - 1$.

VIII. (3) Give an explicit example of a partition of the interval $[0, e]$ that has mesh 10^{-2}. You may use the fact that e is approximately 2.718281828459

Let $x_0 = 0$, $x_1 = 0.01$, $x_2 = 0.02$, and so on (that is, $x_n = n\cdot 0.01$). When we reach $x_{271} = 2.71$, the distance to e is approximately $0.008281828459 < 0.01$. So putting $x_{272} = e$ gives a partition whose mesh is equal to the larger of 0.01 and $e - 2.71$, that is, 0.01.

IX. (5) Calculate the Riemann sum for the following partition and function, using right-hand endpoints as the sample points x_i^*: the function is $f(x) = x$, the interval is $[0, 2]$, and the partition has $n = 4$ with $x_1 = 0.89$, $x_2 = 1.89$, and $x_3 = 1.99$. Leave the answer as a sum of expressions involving decimal numbers; do not carry out the arithmetic.

Since we are using right-hand endpoints, we have $x_i^* = x_i$. So the Riemann sum is $f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + f(x_4)\Delta x_4 = 0.89 \cdot 0.89 + 1.89 \cdot 1 + 1.99 \cdot 0.1 + 2 \cdot 0.01 = (0.89)^2 + 1.89 + 0.199 + 0.02$.

X. (5) A continuous function $f(x)$ is positive and *increasing* for $0 \le x \le 1$. A partition $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ is selected. Let $L = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x_i$ be the Riemann sum for $f(x)$ computed using *left-hand endpoints* as the sample points, and let $R = \sum_{i=1}^{n} f(x_i)\,\Delta x_i$ be the Riemann sum for $f(x)$ computed using *right-hand endpoints* as the sample points. Using pictures to clarify your explanation, and regarding $\int_0^1 f(x)\,dx$ as the area under $y = f(x)$ between $x = 0$ and $x = 1$, explain why $L < \int_0^1 f(x)\,dx$ an $\int_0^1 f(x)\,dx < R$.

For the left-endpoint picture, all the rectangles are inside the region under the graph $y = f(x)$, so $L < \int_0^1 f(x)\,dx$, while for the right-endpoint picture, the area under the graph is inside the (union of the) rectangles, so $\int_0^1 f(x)\,dx < R$.

XI. (4) State the Fundamental Theorem of Calculus (both parts, of course).

Let f be a continuous function on $[a, b]$. Then $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$. If $F' = f$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

XII. (4) Verify that $y = a\sinh(x) + b\cosh(x)$ is a solution to the differential equation $y'' - y = 0$.

$y' = a\cosh(x) + b\sinh(x)$, so $y'' = a\sinh(x) + b\cosh(x) = y$.

XIII. (8) Use l'Hôpital's rule to calculate the following limits:

1. $\displaystyle\lim_{t\to 0} \frac{5^t - 3^t}{t}$.

$$\lim_{t\to 0} \frac{5^t - 3^t}{t} = \lim_{t\to 0} \frac{\ln(5)5^t - \ln(3)3^t}{1} = \ln(5) - \ln(3).$$

2. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{\pi}{x}\right)^x$.

$\displaystyle\lim_{x\to\infty} \left(1 + \tfrac{\pi}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{\pi}{x}\right)^x\right)} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{\pi}{x}\right)}$. Now $\displaystyle\lim_{x\to\infty} x\ln\left(1 + \tfrac{\pi}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{\pi}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{\pi}{x}} \cdot \frac{-\pi}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\pi}{1 + \frac{\pi}{x}} = \frac{\pi}{1+0} = \pi$, so $\displaystyle\lim_{x\to\infty} e^{x\ln\left(1+\frac{\pi}{x}\right)} = e^{\pi}$.

XIV. (6) For each of the following rational functions, write out the form of the partial fraction decomposition. Do not solve for unknown values of the coefficients.

1. $\dfrac{x^4-x^2}{(x^2+1)^3}$

$$\frac{x^4-x^2}{(x^2+1)^3}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{Ex+F}{(x^2+1)^3}$$

2. $\dfrac{1}{(x+1)^2(x+3)^2(x^2-1)^2}$

$$\frac{1}{(x+1)^2(x+3)^2(x^2-1)^2}=\frac{1}{(x+1)^2(x+3)^2(x+1)^2(x-1)^2}=\frac{1}{(x+3)^2(x-1)^2(x+1)^4}$$
$$=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{x+1}+\frac{F}{(x+1)^2}+\frac{G}{(x+1)^3}+\frac{H}{(x+1)^4}$$

XV. (6) Calculate the derivatives of the following functions:

1. $\displaystyle\int_2^{\ln(x)}\frac{1}{\ln(t)}\,dt.$

$$\frac{d}{dx}\int_2^{\ln(x)}\frac{1}{\ln(t)}\,dt=\frac{1}{\ln(\ln(x))}\cdot\frac{d}{dx}(\ln(x))=\frac{1}{x\ln(\ln(x))}.$$

2. $\displaystyle\int_0^{\int_0^{x^2}e^{u^2}\,du}e^{t^2}\,dt.$

$$\frac{d}{dx}\int_0^{\int_0^{x^2}e^{u^2}\,du}e^{t^2}\,dt=e^{\left(\int_0^{x^2}e^{u^2}\,du\right)^2}\cdot\tfrac{d}{dx}\left(\int_0^{x^2}e^{u^2}\,du\right)=e^{\left(\int_0^{x^2}e^{u^2}\,du\right)^2}\cdot e^{(x^2)^2}\tfrac{d}{dx}\left(x^2\right)=2xe^{x^4}e^{\left(\int_0^{x^2}e^{u^2}\,du\right)^2}.$$

XVI. (12) Let R be the region between $y=0$ and $y=\frac{1}{x}$ for $1\le x<\infty$.

1. Calculate the volume of the solid E obtained when R is rotated about the x-axis (the volume is given by an improper integral, whose value you will need to calculate).

$$V=\int_1^\infty\pi\left(\frac{1}{x}\right)^2dx=\lim_{b\to\infty}\int_1^b\frac{\pi}{x^2}\,dx=\lim_{b\to\infty}\frac{-\pi}{x}\Big|_1^b=\lim_{b\to\infty}\pi-\frac{\pi}{b}=\pi$$

2. Write an improper integral whose value represents the surface area of E (not including the side disk where $x=1$, just the part produced by rotating $y=\frac{1}{x}$).

$$ds=\sqrt{1+\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right)^2}\,dx=\sqrt{1+\frac{1}{x^4}}\,dx\text{, so the length is }\int_1^\infty 2\pi\cdot\frac{1}{x}\cdot\sqrt{1+\frac{1}{x^4}}\,dx.$$

3. By making a comparison, verify that the integral which represents the surface area of E diverges to ∞.

For $x\ge1$, $2\pi\dfrac{1}{x}\sqrt{1+\dfrac{1}{x^4}}\ge2\pi\dfrac{1}{x}\sqrt{1}=\dfrac{2\pi}{x}\ge0$. Since $\displaystyle\int_1^\infty\frac{2\pi}{x}\,dx=\lim_{b\to\infty}\int_1^b\frac{2\pi}{x}\,dx=\lim_{b\to\infty}2\pi\ln(x)\Big|_1^b=\lim_{b\to\infty}2\pi\ln(b)=\infty$, the original integral for the surface area also diverges.

XVII. Use integration by parts to verify that $f(a+h) - f(a) - f'(a)h = \displaystyle\int_0^h (h-t) f''(a+t)\,dt$.
(6)

$$\int_0^h (h-t)\, f''(a+t)\,dt = (h-t)\, f'(a+t)\Big|_0^h + \int_0^h f'(a+t)\,dt$$
$$= -f'(a)h + f(a+t)\Big|_0^h = -f'(a)h + f(a+h) - f(a)$$