I. Use integration to find a function \( f(x, y, z) \) for which \( \nabla f = (y + z) \mathbf{i} + (x + z) \mathbf{j} + (x + y) \mathbf{k} \).

(4)

II. Let \( C \) be the portion of the circle of radius 2 with center at the origin that lies in the first quadrant \( x \geq 0, y \geq 0 \). By direct calculation using a parameterization of \( C \), evaluate the following line integrals.

1. \( \int_C x^2 y \, ds \)

2. \( \int_C xy \, dy \)

3. \( \int_C (x \mathbf{i} + y \mathbf{j}) \cdot d\mathbf{r} \)

(9)

III. Use the Fundamental Theorem of Calculus to carry out a partial calculation of \( \iint_R \frac{\partial Q}{\partial y} \, dA \), where \( R \) is the rectangle \( 1 \leq x \leq 3, 2 \leq y \leq 4 \), and \( Q(x, y) \) is a function of \( x \) and \( y \).

(3)
IV. Use Green’s Theorem to calculate \( \int_C 3xy \, dx + 5x^2y^2 \, dy \), where \( C \) is the triangle with vertices \((0,0), (1,0), \) and \((1,1)\).

V. Calculate the curl and the divergence of the vector field \( x^2 \mathbf{i} + y^2 \mathbf{j} - xyz \mathbf{k} \).

VI. The figure to the right shows a vector field \( \vec{F} = P \mathbf{i} + Q \mathbf{j} \) and three oriented arcs.

1. Near each arc, write a small “+” if the line integral of \( \vec{F} \) along that arc appears to be positive, a “−” if it appears to be negative, and a “0” if it appears to be 0.

2. Does it appear that \( \frac{\partial P}{\partial x} \) is positive, negative, or 0?

3. Does it appear that \( \frac{\partial Q}{\partial y} \) is positive, negative, or 0?

4. Does it appear that \( \text{div}(\vec{F}) \) is positive, negative, or 0?
VII. Let $S$ be the surface given by $x = u \cos(v)$, $y = u \sin(v)$, and $z = u$, where the domain of the parameterization is the rectangle $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

1. Calculate $\vec{r}_u$, $\vec{r}_v$, $\vec{r}_u \times \vec{r}_v$, and $\| \vec{r}_u \times \vec{r}_v \|$.

2. Sketch the domain $R$ in the $uv$-plane. Tell the points in $R$ where locally the parameterization neither stretches nor contracts area.

3. Find an equation in $x$, $y$, and $z$ satisfied by all points in the surface (hint: start by calculating $x^2 + y^2$).

VIII. Let $f(x, y, z) = \sin(x^2 + y^2 + z)$. Let $C_1$ be the line segment from $(0, 0, 0)$ to $(1, 1, 0)$, and let $C_2$ be the curve on the surface $z = e^{xy}$ that lies directly above $C_1$. Calculate $\int_{C_1} \nabla f \cdot d\vec{r}$ and $\int_{C_2} \nabla f \cdot d\vec{r}$.
IX. Let $C$ be the unit circle in the $xy$-plane and let $\vec{T}$ be its unit tangent vector. Suppose that a certain vector field $\vec{F}$ has the property that each point $(x, y)$ in $C$, $\vec{F} \cdot \vec{T} = \pi$. Find $\int_C \vec{F} \cdot d\vec{r}$.

X. The figure below shows four regions in the plane. Below each region, write a very small letter $m$ if the region is simply connected, and a very small letter $n$ if the region is not simply-connected. The three dots on the last region means that the region continues to the right forever.

XI. Find a vector field $\vec{F}$ in the plane so that if $C$ is any path which does not pass through the origin, and $C$ starts at $P$ and ends at $Q$, then $\int_C \vec{F} \cdot d\vec{r}$ equals the distance from $Q$ to the origin, minus the distance from $P$ to the origin.

XII. Give an example of a 2-dimensional vector field $P\vec{i} + Q\vec{j}$ which is not conservative but which does satisfy the condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. You do not need to verify these properties, just write down the vector field.