Mathematics 3113-005 Name (please print) ______________________

Final Examination

May 8, 2002

I. (5) Use an integrating factor to solve the first-order linear differential equation $y' + \frac{1}{2x}y = \frac{10}{\sqrt{x}}$, $y(2) = 1$.

II. (3) Give an example of a first-order initial value problem of the form $y' = f(x, y)$, $y(0) = 0$, having nonunique solutions. Give two different solutions and verify that they satisfy the equation.

III. (6) For an n^{th}-order linear differential equation $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$:

1. Tell what it means to say that the equation is *homogeneous.*

2. State the Principle of Superposition.

3. Assuming that $f(x)$ and the coefficient functions $p_1(x), \ldots, p_n(x)$ are continuous, tell the initial conditions, at a point $x = a$, that guarantee that the equation has a unique solution.

IV. (4) Write the differential equation $y^{(6)} + y^{(5)} + y^{(4)} + y^{(3)} + y'' + y' + y = x$ as a system of first-order equations.

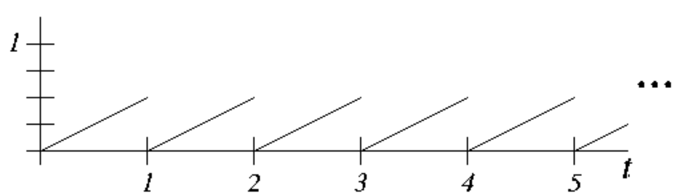
V. (12) Calculate the following Laplace transforms and inverse Laplace transforms, following any special instructions given. Make use of the table of formulas whenever possible.

1. $\mathcal{L}(f(t))$ if $f(t) = t\cosh(t)$

2. $\mathcal{L}(f(t))$ if $f'(t) = \cosh(t)$ and $f(0) = 1$, using the formula for $\mathcal{L}(f'(t))$.

3. $\mathcal{L}^{-1}\left(\dfrac{e^{-2s}}{s^2+2}\right)$

4. $\mathcal{L}(f(t))$, where $f(t)$ is the function shown here:



VI. (6) Solve the nonhomogeneous linear differential equation $y' + \frac{1}{2x}y = \frac{10}{\sqrt{x}}$ using the idea of variation of parameters, as follows.

1. Use the method of separation of variables to find that $y_c(x) = \frac{1}{\sqrt{x}}$ is a solution to the homogeneous equation $y' + \frac{1}{2x}y = 0$.

2. Vary the solution $y_c(x) = \frac{1}{\sqrt{x}}$. That is, let $u = u(x)$ represent an unknown function, and use the function $y = uy_c = u\,\frac{1}{\sqrt{x}}$ as a trial solution in $y' + \frac{1}{2x}y = \frac{10}{\sqrt{x}}$. Obtain an explicit expression for $u(x)$, involving an unknown constant, and use it to find the general solution of the nonhomogeneous equation.

VII. (2) Write a function whose derivative is $\sin(x^2)$.

VIII. (3) Use Laplace transform methods to solve $x' = \delta_2(t)$, $x(0) = 3$.

IX. (4) Let $f(x)$ be a function, all of whose derivatives exist at $x = a$.

1. Write the general formula for the Taylor series of $f(x)$ at $x = a$.

2. Define what it means to say that $f(x)$ is *analytic* at $x = a$.

X. Solve the linear differential equation $y'' + 4y' + 4y = e^{-2x}$ as follows.
(10)

1. Use the characteristic equation to find a general solution to the associated homogeneous equation.

2. Use the method of undetermined coefficients to obtain a trial solution, and put it in the equation to determine a particular solution. (The general formula for the trial solution is $x^s\big((A_0 + A_1x + \cdots + A_mx^m)e^{rx}\cos(kx) + (B_0 + B_1x + \cdots + B_mx^m)e^{rx}\sin(kx)\big)$.)

3. Use the general solution of the associated homogeneous equation and the particular solution that you have found to write down a general solution of $y'' + 4y' + 4y = e^{-2x}$, and find the solution that satisfies the initial condition $y(0) = 0$, $y'(0) = 1$.

XI. Use differential operators and Cramer's rule to find a single differential equation whose solution is the
(5) solution y of the following system, but do *not* proceed further with trying to find y.

$$\begin{aligned} x'' &= y - 4x \\ y'' &= 4x - 8y + \sin(t) \end{aligned}$$

XII. Use Laplace transform methods to solve the following system for y.
(5)

$$\begin{aligned} x'' &= -4x \\ y'' &= \ \ 4x - 8y \end{aligned}$$
$$x(0) = 0, \ \ y(0) = 1, \ \ x'(0) = 0, \ \ y'(0) = 0$$

XIII. Use the power series method to find a general solution to $y' + xy = 0$. Use a well-known series to identify the series solution in terms of familiar elementary functions.
(6)

XIV. For the following boundary value problem, find all *positive* eigenvalues, and an associated eigenfunction for each eigenvalue: $y'' + \lambda y = 0$, $y(0) = 0$, $y'(\pi) = 0$.
(6)