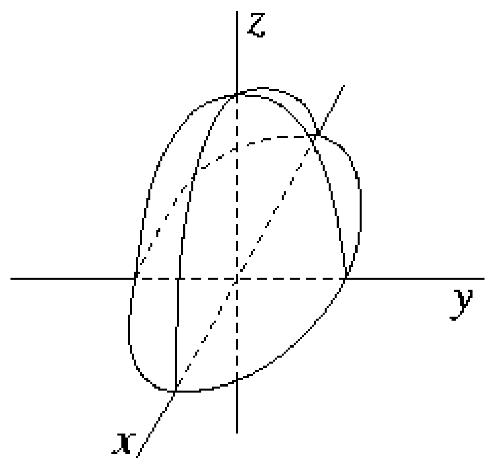
Mathematics 2443-006H Name (please print) ____________________

Examination I

February 21, 2001

Instructions: Find the easier points and do those problems first. Give brief, clear answers.

I. (15) The figure to the right shows the graph of $z = \sqrt{2 - x^2 - 2y^2}$.

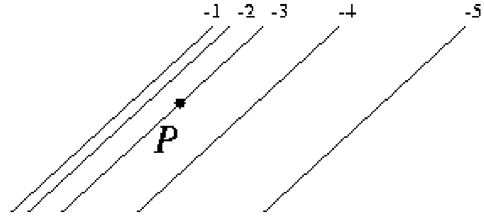


1. Rewrite the defining equation in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Label the values at the five points where the graph intersects one of the coordinate axes.
2. Label the point P on the graph where $x = 1/\sqrt{2}$ and $y = -1/2$.
3. Calculate the vectors $\vec{v}_x$ and $\vec{v}_y$ (the vectors tangent to the graph and having components 1 in the $\vec{\imath}$ direction, for $\vec{v}_x$, or in the $\vec{\jmath}$ direction, for $\vec{v}_y$.)
4. At the point P on the graph, draw the vectors $\vec{v}_x$ and $\vec{v}_y$.
5. Use $\vec{v}_x$ and $\vec{v}_y$ to calculate a normal vector to the surface at the point P.

II. (15) Calculate the following partial derivatives.

1. $\dfrac{dg}{dx}$ if $g(t_1, \ldots, t_n) = 2\sqrt{t_1 + t_2^2 + t_3^3 + \cdots + t_n^n}$ and $\dfrac{dt_i}{dx} = t_i^{i+1}$
2. z_θ if z is a function of x and y, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Noting that $x_\theta = -y$ and $y_\theta = x$, give the answer purely in terms of z_x, z_y, x, and y.
3. $z_{\theta\theta}$ if z is a function of x and y, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Give the answer purely in terms of z_x, z_y, x, and y.

III. (10) The figure to the right shows the level lines for a certain function g near a point P in the xy-plane. Assuming that the level lines give a good guide to the values of g at P, answer the following.



1. Is $\dfrac{\partial g}{\partial x}$ positive, negative, or 0 at P?
2. Is $\dfrac{\partial^2 g}{\partial x^2}$ positive, negative, or 0 at P?
3. Is $\dfrac{\partial^2 g}{\partial x\,\partial y}$ positive, negative, or 0 at P?
4. Draw the gradient of g at P.
5. Draw a direction at P for which the directional derivative is slightly less than 0.

IV. (5) Let $f(x, y) = c$ be a level curve of a differentiable function f. Verify using the chain rule that ∇f is perpendicular to this level curve at each point (start by letting $\gamma(t) = (x(t), y(t))$ be a parameterization of the level curve, and examine $\frac{d}{dt}\big(f(\gamma(t))\big)$).

V. (5) Using implicit differentiation, calculate dR if $\frac{1}{R^2} = \frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2}$.

VI. (10) Calculate the rate of change of $f(x, y) = e^{x^2+y^2}$ at the point $(1, 1)$ in the direction toward $(2, 0)$:

1. Algebraically, using ∇f.
2. Geometrically, by considering level curves.

VII. (5) Partition the interval $0 \leq x \leq 1$ into three intervals with $\Delta x_1 = 0.4$, $\Delta x_2 = 0.1$, and $\Delta x_1 = 0.5$. For the function $f(x) = x^2$, calculate the largest and smallest Riemann sums that can be formed using this partition (the answers are 0.141 and 0.589).

VIII. (5) Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^4}$ does not exist.

IX. (5) Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$ does not exist.

X. (5) Show that $\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^2 + y^4} = 0$ by using an estimate of $\left|\frac{xy^5}{x^2 + y^4}\right|$.

XI. (5) Calculate an equation for the tangent plane to the surface $e^{yz} = e^x$ at the point $(1, 1, 2)$. (Express the surface as a level surface for a certain function of three variables. Do not bother to simplify the equation of the plane.)

XII. (5) Let D be the region $\{(x, y) \mid x^2 + y^2 \leq 1, y \leq 0\}$ in the xy-plane, and consider an integral $\iint_D f(x, y)\, dA$ over the region D.

1. Supply limits for integrating first with respect to x and then with respect to y.
2. Supply limits for integrating first with respect to y and then with respect to x.

XIII. ($1) Bonus: Calculate $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\sin(2 + j/n)}{n}$.