Mathematics 2443-006H
Examination I
February 21, 2001

Instructions: Find the easier points and do those problems first. Give brief, clear answers.

I. The figure to the right shows the graph of \( z = \sqrt{2 - x^2 - 2y^2} \).

1. Rewrite the defining equation in the form \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \).
   Label the values at the five points where the graph intersects one of the coordinate axes.

2. Label the point \( P \) on the graph where \( x = 1/\sqrt{2} \) and \( y = -1/2 \).

3. Calculate the vectors \( \vec{v}_x \) and \( \vec{v}_y \) (the vectors tangent to the graph and having components 1 in the \( \vec{i} \) direction, for \( \vec{v}_x \), or in the \( \vec{j} \) direction, for \( \vec{v}_y \).)

4. At the point \( P \) on the graph, draw the vectors \( \vec{v}_x \) and \( \vec{v}_y \).

5. Use \( \vec{v}_x \) and \( \vec{v}_y \) to calculate a normal vector to the surface at the point \( P \).

II. Calculate the following partial derivatives.

1. \( \frac{dg}{dx} \) if \( g(t_1, \ldots, t_n) = 2\sqrt{t_1^2 + t_2^2 + t_3^2 + \cdots + t_n^2} \) and \( \frac{dt_i}{dx} = t_i^{i+1} \).

2. \( z_\theta \) if \( z \) is a function of \( x \) and \( y \), where \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \). Noting that \( x_\theta = -y \) and \( y_\theta = x \), give the answer purely in terms of \( z_x, z_y, x, \) and \( y \).

3. \( z_{\theta\theta} \) if \( z \) is a function of \( x \) and \( y \), where \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \). Give the answer purely in terms of \( z_x, z_y, x, \) and \( y \).

III. The figure to the right shows the level lines for a certain function \( g \) near a point \( P \) in the \( xy \)-plane. Assuming that the level lines give a good guide to the values of \( g \) at \( P \), answer the following.

1. Is \( \frac{\partial g}{\partial x} \) positive, negative, or 0 at \( P \)?

2. Is \( \frac{\partial^2 g}{\partial x^2} \) positive, negative, or 0 at \( P \)?

3. Is \( \frac{\partial^2 g}{\partial x \partial y} \) positive, negative, or 0 at \( P \)?

4. Draw the gradient of \( g \) at \( P \).

5. Draw a direction at \( P \) for which the directional derivative is slightly less than 0.
IV. Let \( f(x,y) = c \) be a level curve of a differentiable function \( f \). Verify using the chain rule that \( \nabla f \) is perpendicular to this level curve at each point (start by letting \( \gamma(t) = (x(t), y(t)) \) be a parameterization of the level curve, and examine \( \frac{d}{dt}(f(\gamma(t))) \)).

V. Using implicit differentiation, calculate \( dR \) if \( \frac{1}{R^2} = \frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2} \).

VI. Calculate the rate of change of \( f(x,y) = e^{x^2+y^2} \) at the point \( (1,1) \) in the direction toward \( (2,0) \):

1. Algebraically, using \( \nabla f \).
2. Geometrically, by considering level curves.

VII. Partition the interval \( 0 \leq x \leq 1 \) into three intervals with \( \Delta x_1 = 0.4 \), \( \Delta x_2 = 0.1 \), and \( \Delta x_1 = 0.5 \). For the function \( f(x) = x^2 \), calculate the largest and smallest Riemann sums that can be formed using this partition (the answers are 0.141 and 0.589).

VIII. Show that \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^4} \) does not exist.

IX. Show that \( \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} \) does not exist.

X. Show that \( \lim_{(x,y) \to (0,0)} \frac{xy^5}{x^2 + y^4} = 0 \) by using an estimate of \( \left| \frac{xy^5}{x^2 + y^4} \right| \).

XI. Calculate an equation for the tangent plane to the surface \( e^{yz} = e^x \) at the point \( (1,1,2) \). (Express the surface as a level surface for a certain function of three variables. Do not bother to simplify the equation of the plane.)

XII. Let \( D \) be the region \( \{ (x,y) \mid x^2 + y^2 \leq 1, y \leq 0 \} \) in the \( xy \)-plane, and consider an integral \( \iint_D f(x,y) \, dA \) over the region \( D \).

1. Supply limits for integrating first with respect to \( x \) and then with respect to \( y \).
2. Supply limits for integrating first with respect to \( y \) and then with respect to \( x \).

XIII. Bonus: Calculate \( \lim_{n \to \infty} \sum_{j=1}^{n} \frac{\sin(2 + j/n)}{n} \).