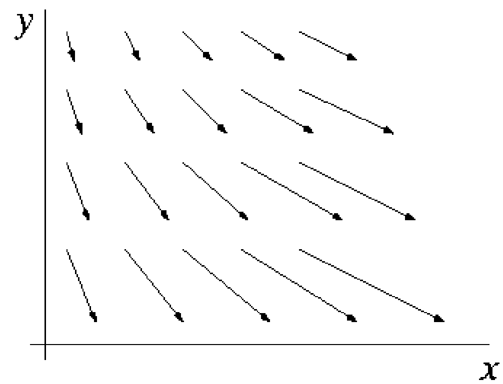
Mathematics 2443-006H Name (please print) ____________________

Final Examination

May 10, 2002

I. (6) Draw two coordinate systems, and make good sketches of the vector fields $-x\vec{\imath}+\vec{\jmath}$ and $\dfrac{-y}{x^2+y^2}\vec{\imath}+\dfrac{x}{x^2+y^2}\vec{\jmath}$.

II. (8) The figure to the right shows a vector field $P(x,y)\vec{\imath}+Q(x,y)\vec{\jmath}$ in the xy-plane. Let $\vec{F}$ be the vector field $P\vec{\imath}+Q\vec{\jmath}+0\,\vec{k}$. Answer the following questions, based on the most probable structure for $\vec{F}$ as indicated in the figure.



1. Determine which of P_x, P_y, Q_x, and Q_y are positive, negative, or 0.
2. Say what you can about $\mathrm{curl}(\vec{F})$.
3. Say what you can about $\mathrm{div}(\vec{F})$.

III. (8) State Green's theorem. Prove it for the special case of $\int_C (P\vec{\imath}+Q\vec{\jmath})\cdot d\vec{r}$, where C is the boundary of the unit square $0\le x\le 1$, $0\le y\le 1$ (to save time, you can examine the line integral carefully on one of the four sides, and say that the others behave similarly).

IV. (10) A certain surface S which is a cone is parameterized by $x=2v\cos(u)$, $y=2v\sin(u)$, and $z=v$, where $0\le u\le 2\pi$ and $v\ge 0$.

1. Find an xyz-equation for S, and use it to sketch S.
2. Compute the vectors $\vec{r}_u$, $\vec{r}_v$, and $\vec{r}_u\times\vec{r}_v$. Sketch them at some typical point on the surface.
3. Calculate $\|\,\vec{r}_u\times\vec{r}_v\,\|$. Use it to calculate the surface area of the portion of S that lies above the unit disc in the xy-plane.

V. (12) Let a be a positive constant, and let S be the sphere of radius a centered at the origin of xyz-space. Parameterize S by $x=a\cos(\theta)\sin(\phi)$, $y=a\sin(\theta)\sin(\phi)$, $z=a\cos(\phi)$, so that $dS=a^2\sin(\phi)\,dR$.

1. Use the parameterization to calculate that the area of S is $4\pi a^2$.
2. Calculate $\iint_S z^2\,dS$.
3. Calculate $\iint_S z\vec{k}\cdot d\vec{S}$, using the definition of the surface integral of a vector field, and making use of your calculated value of $\iint_S z^2\,dS$ (write β for this value, if you could not complete the previous item). Hint: What is $\vec{n}$?
4. Calculate $\iint_S z\vec{k}\cdot d\vec{S}$, using the Divergence Theorem.

VI. (8) A region S in the xy-plane is parameterized by the equations $x=2u+3v$, $y=3u-2v$, where the parameter domain is the unit square R given by $0\le u\le 1$, $0\le v\le 1$.

1. Draw an xy-coordinate system and sketch S. Hint: figure out where the four corners and the four sides are sent.
2. Use a change of coordinates and the Jacobian to calculate $\iint_S x-y\,dS$.

VII. (8) Apply Stokes' Theorem on the surface S given by $x^2 + y^2 + z^2 = 1$ and $z \geq 0$ to calculate $\int_C \left(\frac{-y}{x^2+y^2}\vec{\imath} + \frac{x}{x^2+y^2}\vec{\jmath} \right) \cdot d\vec{r}$, where C is the unit circle. Hints: 1. note first that this integral equals $\int_C (-y\vec{\imath} + x\vec{\jmath}) \cdot d\vec{r}$ (why is this true?), then apply Stokes' Theorem to this line integral. 2. if you can, use the interpretation of the surface integral as the flow across S per unit time to find the value of the surface integral without using direct computation.

VIII. (6) Use Green's Theorem and part 1 of the hint of problem VII to calculate $\int_C \left(\frac{-y}{x^2+y^2}\vec{\imath} + \frac{x}{x^2+y^2}\vec{\jmath} \right) \cdot d\vec{r}$, where C is the unit circle.

IX. (6) Find a path C in 3-dimensional space that starts at the origin and ends at a point with at least two integer coordinates, and for which $\int_C (4xe^z\vec{\imath} + \cos(y)\vec{\jmath} + 2x^2e^z\vec{k}) \cdot d\vec{r} = \frac{2}{e} - \frac{1}{\sqrt{2}}$.

X. (6) Find a unit vector $\vec{u}$ such that the directional derivative of the function $f(x, y, z) = e^{x^2yz}$ at $(1, 2, 3)$ in the direction of $\vec{u}$ equals $\frac{5e^6}{\sqrt{2}}$.

XI. (6) Use the Divergence Theorem to calculate $\iint_S (x^3\vec{\imath} + y^3\vec{\jmath} + z^3\vec{k}) \cdot d\vec{S}$ where S is the surface of the portion of the unit ball $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ that lies in the first octant, and has the outward normal.

XII. (6) Define what it means to say that a planar domain D is *simply-connected.* Give explicit examples (drawing good pictures is explicit enough) of each of the following.

1. Two planar domains D_1 and D_2 such that D_1 and D_2 are simply-connected, but $D_1 \cup D_2$ is not simply-connected.
2. Two planar domains D_1 and D_2 such that D_1 and D_2 are not simply-connected, but $D_1 \cup D_2$ is simply-connected.