I. The functions $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$ are solutions to the differential equations $y'' + 4y = 0$. (13)

1. Calculate the Wronskian $W(\cos(2x), \sin(2x))$.

2. Find a solution $y$ of the differential equation $y'' + 4y = 0$ that satisfies the initial conditions $y(0) = 3$ and $y'(0) = 8$.

3. Given that the function $\frac{e^x}{5}$ satisfies the differential equation $y'' + 4y = e^x$, write a general solution of the differential equation $y'' + 4y = e^x$. 

II. Using the formula \(x^s((A_0 + A_1x + \cdots + A_mx^m)e^{rx} \cos(kx) + (B_0 + B_1x + \cdots + B_mx^m)e^{rx} \sin(kx))\), write trial solutions for the method of undetermined coefficients for the following differential equations, but do not substitute them into the equations or proceed further with finding the solution.

1. \(y'' + y = x \cos(x)\)

2. \(y^{(3)} + 3y'' + 3y' + y = xe^{-x}\) (Fact: \(\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3\))

III. Rewrite \(2 \cos(7x) - 11 \sin(7x)\) in phase-angle form. Give the exact function (so your answer will involve the inverse tangent function).
IV. Verify that the functions \((x + 1)^2, x^2, x,\) and 1 are linearly dependent on the interval of all real numbers. (5)

V. Making use of the equations \(u'_1 y_1 + u'_2 y_2 = 0,\ u'_1 y'_1 + u'_2 y'_2 = f(x),\) and the integration formula \(\int \sin^2(x) \, dx = \frac{1}{2} x - \frac{1}{2} \sin(x) \cos(x),\) and the fact that \(y = c_1 \cos(x) + c_2 \sin(x)\) is a general solution of the homogeneous differential equation \(y'' + y = 0,\) apply the method of variation of parameters to find a particular solution of \(y'' + y = \sin(x).\) (Hint: You will know if you are on the right track if you find that \(u'_2 = \sin(x) \cos(x).\) Then, \(u_2 = \int \sin(x) \cos(x) \, dx = \sin^2(x)/2.\)
VI. Consider the boundary value problem \( y'' + \lambda y = 0; \ y'(0) = 0, \ y(1) = 0. \)
(14)

1. Define what it means to say that a number \( \lambda_i \) is an *eigenvalue* for the boundary value problem. Define what it means to say that a function is an *eigenfunction associated to* \( \lambda_i \).

2. Complete the following argument which shows that this boundary value problem has no negative eigenvalues.

Write \( \lambda = -\alpha^2 \) with \( \alpha > 0 \). The characteristic equation is \( r^2 - \alpha^2 = 0 \), with roots \( \pm \alpha \), so the general solution is \( y = c_1 e^{\alpha x} + c_2 e^{-\alpha x} \). We calculate that \( y' = \alpha c_1 e^{\alpha x} - \alpha c_2 e^{-\alpha x} \).

3. Complete the following argument to find all positive eigenvalues of this boundary value problem, and associated eigenfunctions.

Write \( \lambda = \alpha^2 \) with \( \alpha > 0 \). The characteristic equation is \( r^2 + \alpha^2 = 0 \), with roots \( \pm \alpha i \), so the general solution is \( y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \). We calculate that \( y' = -\alpha c_1 \sin(\alpha x) + \alpha c_2 \cos(\alpha x) \).