Mathematics 4433-001

Examination II

March 20, 2000

Instructions: Give brief and to-the-point answers (do not make the exam longer than it is). Make use of the Riemann-Lebesgue Theorem whenever possible.

I. (12) (a) State the Mean Value Theorem.

For parts (b) and (c), suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function satisfying the hypotheses of the Mean Value Theorem.

(b) Show that if there exists a number M such that $|f'(x)| \le M$ for all $x \in \mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.

(c) Show that if $f'(c) = 0$ for all $c \in [a, b]$, then f is constant on $[a, b]$ (hint: apply the MVT on $[a, x]$ for each $x \in (a, b]$).

II. (14) Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function and let X and Y be partitions of $[a, b]$.

(a) If $X = \{x_0, x_1, \ldots, x_n\}$, define $m_i(f)$, Δx_i, and $\underline{S}(f; X)$.

(b) What can be said about the relation between $\overline{S}(f; X)$ and $\underline{S}(f : Y)$?

(c) If X refines Y, what can be said about the relation between $\underline{S}(f; X)$ and $\underline{S}(f : Y)$?

(d) Define $\underline{S}(f)$ and $\overline{S}(f)$. Define what it means to say that f is Riemann integrable. Assuming that f is Riemann integrable, what is the definition of $\int_a^b f$?

III. (10) Let $f\colon [0, 1] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ if $0 < x \le 1$.

(a) Sketch the graph of f.

(b) Use the definition of $f'(c)$ as a limit to verify that $f'(0)$ exists and determine its value (you may use either the Squeeze Theorem for Limits or the ϵ-δ methodology to calculate the limit).

(c) Is f Riemann integrable on $[0, 1]$? Why or why not?

IV. (5) Without giving any verifications, tell an example of a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ so that $f'(0) = 1$, but f is not increasing on any open interval that contains 0.

V. (6) Give an explicit partition P with $n = 5$ (i. e. P is of the form $\{x_0, x_1, x_2, x_3, x_4, x_5\}$) of the interval $[0, 3]$ with $\| P \| = 1.7$.

VI. (8) Suppose that A and B are bounded subsets of $\mathbb{R}$. Define $A + B$ to be $\{a + b \mid a \in A, b \in B\}$.

(a) Show that $\sup(A) + \sup(B)$ is an upper bound for $A + B$.

(b) Show (making use of known basic facts about sup) that if $\epsilon > 0$ then $\sup(A) + \sup(B) - \epsilon$ is not an upper bound for $A + B$.

VII. (9) Let $f\colon [0, 1] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = 1/n$, if $\dfrac{1}{n+1} < x \le \dfrac{1}{n}$ for $n = 1, 2, \ldots$.

(a) Sketch the graph of f.

(b) Without giving proof, determine $\{x \in [0, 1] \mid f \text{ is not continuous at } x\}$.

(c) Is f Riemann integrable on $[0, 1]$? Why or why not?