Mathematics 2934-010 Name (please print) ____________________

Exam I

September 19, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. Do not simplify the answers unless you find yourself with enough time at the end.

I. Let L be the line through the points $(1,1,0)$ and $(3,0,2)$.
(10)

(i) Find a direction vector for L.

One choice is the vector from $(1,1,0)$ to $(3,0,2)$, which is $2\vec{\imath} - \vec{\jmath} + 2\vec{k}$.

(ii) Write an equation for L as a vector-valued function of t.

$\vec{r}(t) = (\vec{\imath} + \vec{\jmath}) + t(2\vec{\imath} - \vec{\jmath} + 2\vec{k}) = (1+2t)\vec{\imath} + (1-t)\vec{\jmath} + 2t\vec{k}$.

(iii) Write parametric equations for L.

$x = 1 + 2t$, $y = 1 - t$, $z = 2t$.

(iv) Write symmetric equations in x, y, and z for L.

$$\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z}{2}$$

(v) Find the point where L intersects the plane through the origin with normal vector $\vec{\imath} + \vec{\jmath} - \vec{k}$.

The plane has equation $x + y - z = 0$. Points on L have the form $(1+2t, 1-t, 2t)$, so lie on the plane exactly when $(1+2t) + (1-t) - 2t = 0$, that is, when $t = 2$, so the point is $(5, -1, 4)$.

II. Calculate $(a\vec{\imath} + b\vec{\jmath} + c\vec{k}) \times (\vec{\imath} + \vec{\jmath} + \vec{k})$.
(3)

$$\det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} = \vec{\imath}(b \cdot 1 - c \cdot 1) - \vec{\jmath}(a \cdot 1 - c \cdot 1) + \vec{k}(a \cdot 1 - b \cdot 1) = (b-c)\vec{\imath} - (a-c)\vec{\jmath} + (a-b)\vec{k}\ .$$

III. In a reasonably large xy-coordinate system, sketch the vectors $\vec{v} = \sqrt{3}\vec{\imath} + \vec{\jmath}$ and $\vec{w} = -4\vec{\jmath}$.
(10)

(a) Sketch the vector projection of $\vec{v}$ onto $\vec{w}$, and the vector projection of $\vec{w}$ onto $\vec{v}$ (indicate which is which).

(b) Does $\vec{v} \times \vec{w}$ point toward you, or away from you?

Away.

(c) The angle bewteen $\vec{v}$ and $\vec{w}$ is $2\pi/3$ (you do not need to explain this or check it). Use a formula to calculate $\|\vec{v} \times \vec{w}\|$.

$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\ \|\vec{w}\|\ \sin(2\pi/3) = 2 \cdot 4 \cdot (\frac{\sqrt{3}}{2}) = 4\sqrt{3}$.

(d) Tell how you could do part (c) geometrically, without using a formula or even knowing the angle between $\vec{v}$ and $\vec{w}$.

$\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$. Since the parallelogram has base of length 4 and height $\sqrt{3}$, its area is $4\sqrt{3}$.

IV. Four very short problems:
(8) (a) Write an equation for the plane perpendicular to the y-axis and containing the point $(-1, 6, 13)$.

$y = 6$

(b) Write an equation for the sphere of radius $\sqrt{3}$ and center $(-1, 6, 13)$ (do not simplify it).

$(x + 1)^2 + (y - 6)^2 + (z - 13)^2 = 3.$

(c) Write the scalar projection of $a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ onto $\vec{k}$ (preferably not by calculating).

c

(d) Write the vector projection of $a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ onto $\vec{k}$ (preferably not by calculating).

$c\vec{k}.$

V. Use the formula $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$ to find the distance from $(2, 1, -1)$ to the plane $x - y + \sqrt{2}\, z = 1$.
(4)

We have $(a, b, c, d) = (1, -1, \sqrt{2}, -1)$ and $(x_1, y_1, z_1) = (2, 1, -1)$, giving the distance to be $\dfrac{|2 - 1 - \sqrt{2} - 1|}{\sqrt{1 + 1 + 2}} = \dfrac{|-\sqrt{2}|}{2} = \dfrac{1}{\sqrt{2}}$.

VI. By completing the square and other algebraic manipulations, find a substitution of the form $X = x - x_0$,
(6) $Y = y - y_0$, $Z = z - z_0$ that will put the equation $x^2 + 3y^2 - z^2 + 2x - 6y = 5$ into the form $AX^2 + BY^2 + CZ^2 = J$. Put this into one of the standard forms and determine (using your reference sheet) what type of surface it describes (ellipsoid, hyperboloid with one sheet, or hyperboloid with two sheets), but do not proceed further with analyzing it or trying to sketch it.

$$x^2 + 3y^2 - z^2 + 2x - 6y = 5$$
$$x^2 + 2x + 1 + 3(y^2 - 2y + 1) - z^2 = 5 + 1 + 3$$
$$(x + 1)^2 + 3(y - 1)^2 - z^2 = 9$$

Letting $X = x + 1$, $Y = y - 1$, and $Z = z$ puts the equation into the form $X^2 + 3Y^2 - Z^2 = 9$. In the standard form

$$\frac{X^3}{3^2} + \frac{Y^2}{(\sqrt{3})^2} - \frac{Z^2}{3^2} = 1 \ ,$$

there is one minus sign, so the equation describes a hyperboloid with one sheet.

VII. Write equations for the traces of the hyperbolic paraboloid ("saddle surface") $y = 4x^2 - 9z^2$ in the planes
(7) $y = k$, and for $k > 0$ put them into the standard form and draw their graphs in the xz-plane (including finding their asymptotes and intercepts). Do not proceed further with analyzing the equation or trying to sketch its graph.

The equations of the traces are $k = 4x^2 - 9z^2$, or in standard form

$$\frac{x^2}{\left(\frac{\sqrt{k}}{2}\right)^2} - \frac{z^2}{\left(\frac{\sqrt{k}}{3}\right)^2} = 1 \ .$$

This is a hyperbola in the xz-plane. It opens left-and-right since the x^2 term has the plus sign. Its x-intercepts are $\pm\frac{\sqrt{k}}{2}$, and the asymptotes are $z = \pm\frac{\sqrt{k}/3}{\sqrt{k}/2}\, x$ or $z = \pm\frac{2}{3}\, x$.

VIII. For the vectors $\vec{v}_b = 2\vec{\imath} + b\vec{\jmath} - \vec{k}$, find all choices of b such that $\vec{v}_b$ and $-\vec{\imath} + b\vec{\jmath} + b\vec{k}$ meet at right angles.
(3)

We need $0 = (2\vec{\imath} + b\vec{\jmath} - \vec{k}) \cdot (-\vec{\imath} + b\vec{\jmath} + b\vec{k}) = b^2 - b - 2$, which occurs for $b = 1$ or $b = -2$.