Mathematics 2934-010 Name (please print) ______________________

Quiz 6

December 1, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. Remember that ∇f and grad(f) both mean the gradient of f.

I. State Green's Theorem for a domain D with boundary a positively oriented simple loop C. Include any
(2) important conditions that the functions P and Q in its statement must satisfy.

Let P and Q have continuous derivatives on all of D. Then $\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$.

II. Let C be the unit circle, oriented counterclockwise. Use Green's Theorem to calculate
(3) $\displaystyle\int ((2x+y^2)\,\vec{\imath} + (2xy - 2x + \sin(y^2))\,\vec{\jmath}) \cdot d\vec{r}$.

$$\int ((2x+y^2)\vec{\imath} + (2xy - 2x + \sin(y^2)\vec{\jmath}) \cdot d\vec{r} = \iint_D \frac{\partial}{\partial x}(2xy - 2x + \sin(y^2))$$
$$-\frac{\partial}{\partial y}(2x+y^2)\,dA = \iint_D 2y - 2 - 2y\,dA = \iint_D -2\,dA = -2\,\mathrm{area}(D) = -2\pi\ .$$

III. Let T be the triangle with vertices at $(1,0)$, $(0,1)$, and $(0,0)$. Let C be the boundary of T, with the positive
(5) orientation.

(a) Use Green's Theorem to verify that $\displaystyle\int_C x\,dy$ equals the area of T.

$$\int_C x\,dy = \iint_T \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}(0)\,dA = \iint_T 1\,dA = \mathrm{area}(T)\ .$$

(b) Let C_1 be the side of T from $(0,1)$ to $(0,0)$, C_2 be the side of T from $(0,0)$ to $(1,0)$, and C_3 be the side of T from $(1,0)$ to $(0,1)$. Without calculating them, explain why $\displaystyle\int_{C_1} x\,dy$ and $\displaystyle\int_{C_2} x\,dy$ are zero.

On C_1 the integrand x is 0. On C_2, dy is zero.

(c) Use the parameterization $x = 1-t$, $y = t$, $0 \le t \le 1$ to calculate $\displaystyle\int_{C_3} x\,dy$ (from parts (a) and (b), the answer should equal the area of T, that is, $\frac{1}{2}$).

(Notice that t going from 0 to 1 corresponds to traveling on C_3 from $(1,0)$ to $(0,1)$, which is consistent with the positive orientation on C_3.) We have

$$\int_{C_3} x\,dy = \int_0^1 (1-t)\,dt = -\frac{(1-t)^2}{2}\Big|_0^1 = 0 - (-\frac{1^2}{2}) = \frac{1}{2}\ .$$

IV. (3) (a) Define what it means to say that a domain D in the plane is *simply-connected.*

It is *path-connected,* which means that any two paths in D are connected by a path in D, and every simple loop in D must enclose only points of D.

(b) Tell the important and non-obvious property we have studied that is true for vector fields $P\vec{\imath} + Q\vec{\jmath}$ on simply-connected planar domains, but not necessarily true for domains that are not simply-connected.

$P\vec{\imath} + Q\vec{\jmath}$ is a gradient exactly when $P_y = Q_x$.

(c) Give an example of a vector field illustrating that the property in (b) is not necessarily true for non-simply-connected domains. You do not need to verify the properties, just tell the vector field and the domain.

The domain is the plane minus the origin, and the vector field is $\dfrac{-y}{x^2+y^2}\vec{\imath} + \dfrac{x}{x^2+y^2}$.

V. (3) Let f be a scalar field in $\mathbb{R}^3$ (that is, a function of (x, y, z)) and let $\vec{F}$ be a vector field in $\mathbb{R}^3$. Determine whether each of the following is a scalar field, a vector field, or meaningless.

(a) $\operatorname{div}(\operatorname{curl}(\operatorname{curl}(\vec{F})))$

scalar field

(b) $\operatorname{div}(\vec{F} \times \operatorname{curl}(\vec{F}))\ \operatorname{grad}(f)$

vector field

(c) $\operatorname{curl}(\vec{F}) \cdot (\operatorname{curl}(\vec{F}) \times \operatorname{curl}(\vec{F}))$

scalar field

VI. (4) Let C be an oriented path in the plane, with unit tangent $\vec{T}$, and let f be a differentiable function in the plane.

(a) What is a simple interpretation of $\nabla f \cdot \vec{T}$?

$\nabla f \cdot \vec{T}$ is the rate of change of f in the direction of C.

(b) How is $\nabla f \cdot \vec{T}$ related to $\int_C \nabla f \cdot d\vec{r}$?

$\int_C \nabla f \cdot d\vec{r} = \int_C \nabla f \cdot \vec{T}\, ds$.

(c) What do (a) and (b) tell us, at least intuitively, about $\int_C \nabla f \cdot d\vec{r}$?

Since the Fundamental Theorem of Calculus says that $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$, integrating the rate of change $\nabla f \cdot \vec{T}$ on C with respect to arclength should give us the total change of f as we travel along C. That is, $\int_C \nabla f \cdot d\vec{r}$ should be $f(\text{endpoint of } C) - f(\text{starting point of } C)$, which is exactly what the Fundamental Theorem of Line Integrals says.