Mathematics 2934-010 Name (please print) ______________________

Final Exam

December 16, 2011

Instructions: Give concise answers, but clearly indicate your reasoning.

I. Calculate the following:
(12)

(a) $\operatorname{curl}(x\,\vec{\imath} + \vec{\jmath} - y\,\vec{k})$

$$\det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 1 & -y \end{bmatrix} = -\vec{\imath}$$

(b) The gradient $\nabla(xy\sin(z))$.

$$\nabla(xy\sin(z)) = \frac{\partial}{\partial x}(xy\sin(z))\,\vec{\imath} + \frac{\partial}{\partial y}(xy\sin(z))\,\vec{\jmath} + \frac{\partial}{\partial z}(xy\sin(z))\,\vec{k} = y\sin(z)\,\vec{\imath} + x\sin(z)\,\vec{\jmath} + xy\cos(z)\,\vec{k}$$

(c) The rate of change of the function $xy\sin(z)$ at $(1, 1, \pi/2)$ in the direction toward the point $(3, 0, \pi/2 + 1)$.

A vector in the direction from $(1, 1, \pi/2)$ to $(3, 0, \pi/2 + 1)$ is $2\vec{\imath} - \vec{\jmath} + \vec{k}$. It has length $\sqrt{6}$, so a unit vector in this direction is $2/\sqrt{6}\,\vec{\imath} - 1/\sqrt{6}\,\vec{\jmath} + 1/\sqrt{6}\,\vec{k}$. At $(1, 1, \pi/2)$, the gradient is

$$\nabla(xy\sin(z))(1, 1, \pi/2) = \vec{\imath} + \vec{\jmath}\,,$$

so the rate of change is

$$\nabla(xy\sin(z))(1, 1, \pi/2) \cdot (2/\sqrt{6}\,\vec{\imath} - 1/\sqrt{6}\,\vec{\jmath} + 1/\sqrt{6}\,\vec{k}) = (\vec{\imath} + \vec{\jmath}) \cdot (2/\sqrt{6}\,\vec{\imath} - 1/\sqrt{6}\,\vec{\jmath} + 1/\sqrt{6}\,\vec{k}) = 1/\sqrt{6}\,.$$

(d) $\displaystyle\int_C \nabla(xy\sin(z)) \cdot d\vec{r}$, where C is the path given parametrically by $x = t^2$, $y = e^{2t+1}$, $z = \pi t^3/2$, $0 \le t \le 1$.

By the Fundamental Theorem for Line Integrals, $\displaystyle\int_C \nabla(xy\sin(z)) \cdot d\vec{r}$ equals the value of $xy\sin(z)$ at the terminal point $(1, e^3, \pi/2)$ minus its value at the initial point $(0, e, 0)$. This works out to be $e^3 - 0 = e^3$.

II. (a) Define what it means to say that a domain D in the plane is *simply-connected.*
(6)

It is *path-connected,* which means that any two paths in D are connected by a path in D, and every simple loop in D must enclose only points of D.

(b) Tell the important and non-obvious property we have studied that is true for vector fields $P\vec{\imath} + Q\vec{\jmath}$ on simply-connected planar domains, but not necessarily true for domains that are not simply-connected.

$P\vec{\imath} + Q\vec{\jmath}$ is a gradient exactly when $P_y = Q_x$.

(c) Give an example of a vector field illustrating that the property in (b) is not necessarily true for non-simply-connected domains. You do not need to verify the properties, just tell the vector field and the domain.

The domain is the plane minus the origin, and the vector field is $\dfrac{-y}{x^2 + y^2}\,\vec{\imath} + \dfrac{x}{x^2 + y^2}\,\vec{\jmath}$.

III. (8) (a) Let S be the part of the plane $x+y+z=1$ that lies in the first octant, regarded as the graph of the function $f(x,y,z)=1-x-y$ over a parameter domain D in the xy-plane. Write a double integral on D whose value is $\iint_S yz\,dS$. You do not need to specify the limits of integration or calculate the value of the integral, but tell the precise domain D.

D is the triangle bounded by $x=0$, $y=0$, and $x+y=1$. The area element dS is

$$dS = \sqrt{1+\left(\frac{\partial}{\partial x}(1-x-y)\right)^2+\left(\frac{\partial}{\partial y}(1-x-y)\right)^2}\,dD = \sqrt{1+(-1)^2+(-1)^2}\,dD = \sqrt{3}\,dD\ ,$$

so $\iint_S yz\,dS = \iint_D \sqrt{3}y(1-x-y)\,dD$.

(b) For the surface S in part (a), write a double integral in x and y on the parameter domain D whose value is $\iint_S (y\,\vec{\imath}+z\,\vec{\jmath}+\vec{k})\cdot d\vec{S}$. Again, you do not need to specify the limits of integration or calculate the value, but do simplify the integrand.

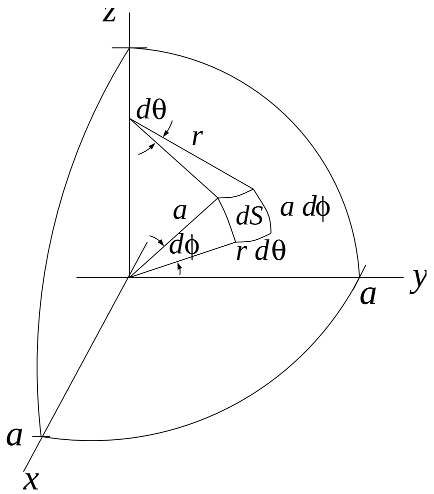
Using the formula $\iint_S (P\,\vec{\imath}+Q\,\vec{\jmath}+R\,\vec{k})\cdot d\vec{S} = \iint_D -P\,g_x - Q\,g_y + R\,dD$, we have

$$\iint_S (y\,\vec{\imath}+z\,\vec{\jmath}+\vec{k})\cdot d\vec{S} = \iint_D (-y)(-1)-(1-x-y)(-1)+1\,dD = \iint_D 2-x\,dD\ .$$

IV. (6) Let D be the domain in the upper half-plane bounded by the loop C which consists of the semicircle $y=\sqrt{2-x^2}$ from $(\sqrt{2},0)$ to $(-\sqrt{2},0)$ and the portion of the x-axis from $(-\sqrt{2},0)$ to $(\sqrt{2},0)$. Use Green's Theorem to express $\int_C (x^2y^3\vec{\imath}-x^3y^2\vec{\jmath})\cdot d\vec{r}$ as a double integral on D. Rewrite the integral in polar coordinates, supplying the limits of integration, but do not go on to compute the actual value of the integral.

$$\int_C (x^2y^3\vec{\imath}-x^3y^2\vec{\jmath})\cdot d\vec{r} = \iint_D -3x^2y^2-3x^2y^2\,dD = \iint_D -6x^2y^2\,dD = \int_0^{\pi}\int_0^{\sqrt{2}} -6r^5\cos^2(\theta)\sin^2(\theta)\,dr\,d\theta.$$

V. (4) Parameterize the sphere S of radius a by the equations $x = a\cos(\theta)\sin(\phi)$, $y = a\sin(\theta)\sin(\phi)$, $z = a\cos(\phi)$. Draw a picture illustrating $d\phi$ and $d\theta$ and the region dS that they span on S. Label various distances and use them to explain the fact that $dS = a^2 \sin(\phi)\, d\theta\, d\phi$.

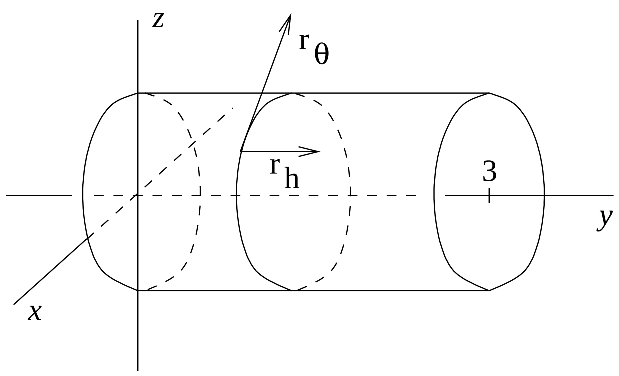


Since $r = a\sin(\phi)$, we have $dS = r\,d\theta\, a\,d\phi = a\sin(\phi)\,d\theta\,a\,d\phi = a^2\sin(\phi)\,d\theta\,d\phi$.

VI. (20) Let S be the portion of the cylinder $x^2 + z^2 = 4$ that lies between the planes $y = 0$ and $y = 3$. Parameterize S as follows:

The parameter domain R is the rectangle in the (θ, h)-plane (so that θ is the horizontal coordinate and h is the vertical coordinate) bounded by $\theta = 0$, $h = 0$, $\theta = 2\pi$, and $h = 3$. The parameterization sends the point (θ, h) in R to the point $(2\cos(\theta), h, 2\sin(\theta))$ in S.

(a) Sketch S in three dimensions. Show typical vectors $\vec{r}_h$ and $\vec{r}_\theta$.



(b) Calculate $\vec{r}_h$ and $\vec{r}_\theta$.

$\vec{r}_h = \vec{\jmath}$, $\vec{r}_\theta = -2\sin(\theta)\vec{\imath} + 2\cos(\theta)\vec{k}$.

(c) Calculate the normal vector $\vec{r}_h \times \vec{r}_\theta$ and its length $\|\vec{r}_h \times \vec{r}_\theta\|$.

$$\vec{r}_h \times \vec{r}_\theta = \det\begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & 1 & 0 \\ -2\sin(\theta) & 0 & 2\cos(\theta) \end{bmatrix} = 2\cos(\theta)\,\vec{\imath} + 2\sin(\theta)\,\vec{k}, \text{ so } \|\vec{r}_h \times \vec{r}_\theta\| = 2.$$

(d) Use the result of part (c) to determine the relationship between the area elements dR and dS.

$dS = \|\vec{r}_h \times \vec{r}_\theta\|\, dR = 2\, dR$.

(e) Calculate $\displaystyle\iint_S y^2\, dS$

$$\iint_S y^2\, dS = \iint_R 2h^2\, dR = \int_0^{2\pi}\int_0^3 2h^2\, dh\, d\theta = \int_0^{2\pi} 18\, d\theta = 36\pi$$

(f) Calculate $\displaystyle\iint_S (x\vec{\imath} + z\vec{k}) \cdot d\vec{S}$, with respect to the outward normal.

$$\begin{aligned}\iint_S (x\vec{\imath} + z\vec{k}) \cdot d\vec{S} &= \iint_R (x\vec{\imath} + z\vec{k}) \cdot (\vec{r}_h \times \vec{r}_\theta)\; dR = \iint_R (2\cos(\theta)\,\vec{\imath} + 2\sin(\theta)\,\vec{k}) \cdot (2\cos(\theta)\,\vec{\imath} + 2\sin(\theta)\,\vec{k})\, dR \\ &= \iint_R 4\cos^2(\theta) + 4\sin^2(\theta)\, dR = \iint_R 4\, dR = 4\,\text{area}(R) = 4 \cdot 3 \cdot 2\pi = 24\pi\end{aligned}$$

VII. (7) Let C be an oriented path in the plane, with unit tangent $\vec{T}$, and let f be a differentiable function in the plane.

(a) What is a simple interpretation of $\nabla f \cdot \vec{T}$?

$\nabla f \cdot \vec{T}$ is the rate of change of f in the direction of C.

(b) How is $\nabla f \cdot \vec{T}$ related to $\int_C \nabla f \cdot d\vec{r}$?

$\int_C \nabla f \cdot d\vec{r} = \int_C \nabla f \cdot \vec{T}\, ds$.

(c) What do (a) and (b) tell us, at least intuitively, about $\int_C \nabla f \cdot d\vec{r}$?

Since the Fundamental Theorem of Calculus says that $\displaystyle\int_a^b f'(x)\, dx = f(b) - f(a)$, integrating the rate of change $\nabla f \cdot \vec{T}$ on C with respect to arclength should give us the total change of f as we travel along C. That is, $\displaystyle\int_C \nabla f \cdot d\vec{r}$ should be $f(\text{endpoint of } C) - f(\text{starting point of } C)$, which is exactly what the Fundamental Theorem of Line Integrals says.

VIII. Parameterize the sphere S of radius a by the equations $x = a\cos(\theta)\sin(\phi)$, $y = a\sin(\theta)\sin(\phi)$, $z = a\cos(\phi)$,
(13) so that $dS = a^2\sin(\phi)\,d\theta\,d\phi$, and parameters in a rectangle R in the (θ,ϕ)-plane.

(a) Take as given the fact that the outward normal $\vec{r}_\phi \times \vec{r}_\theta$ for this parameterization is $a\sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})$. Express $\iint_S (x\vec{\imath} - z\vec{\jmath} + y\vec{k}) \cdot d\vec{S}$ as an integral of a function of θ and ϕ on the domain R. Supply the limits of integration, but do not calculate the value of the integral.

$$\iint_S (x\vec{\imath} - z\vec{\jmath} + y\vec{k}) \cdot d\vec{S} = \int_0^\pi \int_0^{2\pi} (x\vec{\imath} - z\vec{\jmath} + y\vec{k}) \cdot a\sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})\,dA$$

$$= \int_0^\pi \int_0^{2\pi} a\sin(\phi)(x^2 - zy + yz)\,dA = \int_0^\pi \int_0^{2\pi} a^3\cos^2(\theta)\sin^3(\phi)\,dA\ .$$

(b) State the Divergence Theorem (of course the main formula in it is on the formulas list, besides that you will need to describe the setup for the Divergence Theorem— such as what kinds of surfaces it applies to, and what conditions the vector field must satisfy).

Let E be a region with boundary the surface S, and suppose that $\vec{F}$ is a vector field whose component functions have continuous partial derivatives at every point of an open region that contains E. Give S the outward unit normal vector. Then $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\,dV$.

(c) Use the Divergence Theorem to calculate $\iint_S (x\vec{\imath} - z\vec{\jmath} + y\vec{k}) \cdot d\vec{S}$.

Let E be the unit ball in (x,y,z)-space $\iint_S (x\vec{\imath} - z\vec{\jmath} + y\vec{k}) \cdot d\vec{S} = \iiint_E \operatorname{div}(x\vec{\imath} - z\vec{\jmath} + y\vec{k})\,dV = \iiint_E 1\,dV =$ volume$(E) = \dfrac{4\pi a^3}{3}$.

IX. (10) (a) Let S be a surface in $\mathbb{R}^3$ with boundary the loop C, Let $\vec{n}$ be a unit normal to S. Define the *positive orientation* on C.

The positive direction on C is the direction for which the normal vector traveling in that direction on C has S on its left.

(b) State Stokes' Theorem (of course the main formula in it is on the formulas list, besides that you will need to describe the setup for Stokes' Theorem— such as what kinds of surfaces it applies to, what orientation is to be used on its boundary curve, what conditions the vector field must satisfy).

Let S be a surface in space with a chosen unit normal $\vec{n}$, and give the boundary C of S the positive orientation. Let $\vec{F}$ be a vector field [whose component functions have continuous partial derivatives on an open domain containing S]. Then $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$.

(c) Let $\vec{F}(x,y,z) = 2y\cos(z)\,\vec{\imath} + e^x\sin(z)\,\vec{\jmath} + xe^y\,\vec{k}$ and let S be the hemisphere $x^2+y^2+z^2=9$, $z\geq 0$, oriented upward. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$. (You may need the identity $\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$, although its use may be avoided using Green's Theorem.)

The oriented boundary of S is the circle C of radius 3 in the (x,y)-plane, oriented counterclockwise. Using Stokes' Theorem,

$$\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_C (2y\cos(0)\vec{\imath} + e^x\sin(0)\vec{\jmath} + xe^y\vec{k})\cdot d\vec{r} = \int_C (2y\vec{\imath} + xe^y\vec{k})\cdot d\vec{r}\ .$$

Parameterizing the circle as $x = 3\cos(\theta)$, $y = 3\sin(\theta)$, and $z=0$, $0\leq\theta\leq 2\pi$, with velocity vector $-3\sin(\theta)\vec{\imath} + 3\cos(\theta)\vec{\jmath}$, we find

$$\int_C (2y\vec{\imath} + xe^y\vec{k})\cdot d\vec{r} = \int_0^{2\pi} (6\sin(\theta)\vec{\imath} + 3\cos(\theta)e^{3\sin(\theta)}\vec{k})\cdot(-3\sin(\theta)\vec{\imath} + 3\cos(\theta)\vec{\jmath})\,d\theta$$
$$= \int_0^{2\pi} -18\sin^2(\theta)\,d\theta = \int_0^{2\pi} -9 + 9\cos(2\theta)\,d\theta = \int_0^{2\pi} -9\,d\theta = -9\cdot 2\pi = -18\pi\ .$$

A faster way to finish the calculation is to first note that since $\vec{k}$ is perpendicular to the velocity vectors of C, the term $xe^y\vec{k}$ may simply be dropped from $\int_C(2y\vec{\imath} + xe^y\vec{k})\cdot d\vec{r}$. Then, letting D be the disk of radius 3 in the (x,y)-plane and using Green's Theorem, we have

$$\int_C (2y\vec{\imath} + xe^y\vec{k})\cdot d\vec{r} = \int_C 2y\vec{\imath}\cdot d\vec{r} = \int_D \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(2y)\,dD = \int_D -2\,dD = -2\,\text{area}(D) = -2\,\pi\cdot 3^2 = -18\pi\ .$$