Mathematics 2423-001 Name (please print) ______________________

Exam II

October 17, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. It is *not* expected that you will be able to answer all the questions, just do whatever you can in 50 minutes.

I. (12) Evaluate the following integrals.

1. $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$

Letting $u = \pi/x$, $du = -\pi/x^2\,dx$, we have
$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \frac{-1}{\pi}\int \cos(u)\,du = -\sin(u)/\pi + C = -\sin(\pi/x)/\pi + C$$

2. $\displaystyle\int \frac{x+1}{x^2+1}\,dx$

$$\int \frac{x+1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \tfrac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + C$$

3. $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\,dx$

Letting $u = 1-x$, $du = -dx$, we have
$$\int \frac{x^2}{\sqrt{1-x}}\,dx = -\int \frac{(1-u)^2}{\sqrt{u}}\,dx = -\int u^{3/2} - 2u^{1/2} + u^{-1/2}\,dx = -2u^{5/2}/5 + 4u^{3/2}/3 - 2\sqrt{u} + C = -2(1-x)^{5/2}/5 + 4(1-x)^{3/2}/3 - 2\sqrt{1-x} + C$$
The substitution $u = \sqrt{1-x}$ will also work. One then has $x = 1-u^2$ and $du = -\frac{1}{2\sqrt{1-x}}\,dx$, so
$$\int \frac{x^2}{\sqrt{1-x}}\,dx = -2\int (1-u^2)^2\,du = -2\int 1 - 2u^2 + u^4\,du = -2u + 4u^3/3 - 2u^5/5 + C = -2\sqrt{1-x} + 4(1-x)^{3/2}/3 - 2(1-x)^{5/2}/5 + C$$

II. (3) If $f(x)$ is the slope of a trail at a distance x miles from the start of the trail, what does the integral $\displaystyle\int_3^5 f(x)\,dx$ represent?

It represents the difference in altitude between the location at distance 3 and the location at distance 5.

III. (4) By substituting $u = \dfrac{t}{a}$, verify that $\displaystyle\int_a^{ab} \frac{1}{t}\,dt = \ln(b)$.

Putting $u = t/a$, $t = au$, $dt = a\,du$, we have $\displaystyle\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{au}\,a\,du = \int_1^b \frac{1}{u}\,du = \ln(b)$.

IV. (8) Write definite integrals to compute each of the following, but *do not* simplify or evaluate them.

(a) The volume of the solid produced when the region bounded by $y = (x-2)^2$ and $y = 8x - 16$ is rotated about the line $y = -1$.

Solving $8x - 16 = (x-2)^2$ we find that the curves intersect at $x = 2$ and $x = 10$, and consideration of the graphs shows that $8x - 16 > (x-2)^2$ in that range. Taking slices perpendicular to $y = -1$, the cross-section is the region between circles of radii $(x-2)^2 + 1$ and $8x - 15$, so the volume is $\displaystyle\int_2^{10} \pi((8x-15)^2 - ((x-2)^2+1)^2)\,dx$.

Alternatively but more complicated, one may solve the equations for x in terms of y and view the region as lying between $x = 2 + y/8$ and $x = \sqrt{y} + 2$ for $0 \le y \le 64$. Using cylindrical cross-sections, the distance to the axis is $y + 1$ and the volume is $\displaystyle\int_0^{64} 2\pi(y+1)(\sqrt{y} + 2 - (2 + y/8))\,dy$.

(b) The volume of the solid produced when the region in part (a) is rotated about the line $x = -1$.

The distance from x to the axis of rotation is $x + 1$, so taking cylindrical slices we can express the volume as $\displaystyle\int_2^{10} 2\pi(x+1)(8x - 16 - (x-2)^2)\,dx$.

Alternatively, but more complicated, one may use horizontal slices, so the cross-sections lie between circles of radius $\sqrt{y} + 3$ and $3 + y/8$, giving volume $\displaystyle\int_0^{64} \pi((\sqrt{y}+3)^2 - (3 + y/8)^2)\,dy$.

V. (11) (a) Calculate and simplify: $\dfrac{d}{dx}\ln(x + \sqrt{x^2-1})$

$$\frac{d}{dx}\ln(x + \sqrt{x^2-1}) = \frac{1}{x + \sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}}\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

(b) Simplify and calculate: $\dfrac{d}{dz}\ln\left(\sqrt{\dfrac{a^2-z^2}{a^2+z^2}}\right)$

$$\frac{d}{dz}\ln\left(\sqrt{\frac{a^2-z^2}{a^2+z^2}}\right) = \frac{d}{dz}\left(\tfrac{1}{2}\ln(a^2-z^2) - \tfrac{1}{2}\ln(a^2+z^2)\right) = -\frac{z}{a^2-z^2} - \frac{z}{a^2+z^2}$$

(c) Calculate the average value of $\dfrac{1}{1+x^2}$ between $x = 0$ and $x = \sqrt{3}$.

$$\frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \frac{1}{\sqrt{3}}\tan^{-1}(x)\Big|_0^{\sqrt{3}} = \frac{1}{\sqrt{3}}(\tan^{-1}(\sqrt{3}) - \tan^{-1}(0)) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - 0\right) = \frac{\pi}{3\sqrt{3}}$$

VI. (13) Potpourri:

1. Define what it means to say that a function f is *injective.*

   It means that if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

2. For an injective function f with domain A and range B, define the *inverse function* g.

   The domain of g is B, its range is A, and $g(x) = y$ exactly when $f(y) = x$.

3. State the Intermediate Value Theorem.

   If f is a continuous function on a closed interval $[a, b]$, and N is any number between $f(a)$ and $f(b)$, then there exists a c between a and b such that $f(c) = N$.

   (Many people stated the Mean Value Theorem for Integrals.)

4. Show that for any integer $n \geq 2$, $\ln(n) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$.

   Partition the interval $[1, n]$ into $n - 1$ equal subintervals, and draw the picture of the Riemann sum that estimates the area under $y = 1/t$ on this interval using the left endpoints of the intervals as the x_i^*. The rectangles completely enclose the area, since $1/t$ is a decreasing function. The area under $1/t$ is $\ln(n)$ and the Riemann sum is the sum of the areas of the rectangles, which is $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$, giving the inequality.

5. Draw a right triangle with hypotenuse of length 1 and a side of length x. Indicate the length of the third side and correctly label the interior angles as $\sin^{-1}(x)$ and $\cos^{-1}(x)$. Use the triangle to find $\cot(\sin^{-1}(x))$.

   The third side has length $\sqrt{1 - x^2}$. The angle opposite the side labeled x is $\sin^{-1}(x)$ and the angle adjacent to that side is $\cos^{-1}(x)$. From the triangle we read off that $\cot(\sin^{-1}(x)) = \dfrac{\sqrt{1 - x^2}}{x}$.