Mathematics 2423-001 Name (please print) ____________________

Exam I

September 19, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. It is *not* expected that you will be able to answer all the questions, just do whatever you can in 50 minutes.

I. For the function $f(x) = x^2$ on the interval $[-1, 6]$ and the partition $-1 < 2 < 4 < 6$ of $[-1, 6]$:
(6)

(a) Calculate the Riemann sum that uses the midpoints of the subintervals as the x_i^*.

The subintervals are $[-1, 2]$, $[2, 4]$, and $[4, 6]$, with respective midpoints $\frac{1}{2}$, 3, and 5. So the Riemann sum is

$$\frac{1}{2^2} \cdot 3 + 3^2 \cdot 2 + 5^2 \cdot 2 \quad \left(= 68\tfrac{3}{4}\right) .$$

(b) Calculate the smallest Riemann sum.

On the three subintervals, the minimum values of x^2 occur at 0, 2, and 4 respectively. So the smallest Riemann sum is

$$0^2 \cdot 3 + 2^2 \cdot 2 + 4^2 \cdot 2 \;\; (= 40) .$$

II. (a) State the definition we used for the *rate of change* of a function f at a point a (in the definition, use
(9) m for the rate of change).

The number m is the *rate of change* of f at a if $f(a + h) = f(a) + mh + E(h)$ with $\lim\limits_{h\to 0} \dfrac{E(h)}{h} = 0$.

(b) Use our definition to verify that the rate of change of $x^3 + 8x$ at 0 is 8. That is, taking this function as $f(x)$, $a = 0$, and 8 as the rate of change, write $f(0 + h) = h^3 + 8h$ as an expression involving the rate of change and the error $E(h)$, then analyze $E(h)$ to verify that 8 is the correct rate of change.

Writing $h^3 + 8h = (0^3 + 8 \cdot 0) + 8h + h^3$ shows that when $m = 8$, the error is $E(h) = h^3$. Since $\lim\limits_{h\to 0} \dfrac{E(h)}{h} = \lim\limits_{h\to 0} \dfrac{h^3}{h} = \lim\limits_{h\to 0} h^2 = 0$, this shows that 8 is the rate of change.

(c) Use our definition to verify that the rate of change of $x^3 + 8x$ at 0 is not -2.

Writing $h^3 + 8h = (0^3 + 8 \cdot 0) + (-2)h + (h^3 + 10h)$ shows that when $m = -2$, the error is $E(h) = h^3 + 10h$. Since $\lim\limits_{h\to 0} \dfrac{E(h)}{h} = \lim\limits_{h\to 0} \dfrac{h^3 + 10h}{h} = \lim\limits_{h\to 0} h^2 + 10 = 10 \neq 0$, this shows that -2 is not the rate of change.

III. Without worrying about the hypotheses, state both parts of the Fundamental Theorem of Caclulus.
(6)

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x), \text{ and if } F' = f \text{ then } \int_a^b f(x)\, dx = F(b) - F(a).$$

IV. Use the Mean Value Theorem to verify that if x and y are any two numbers greater than 1, then
(5) $|\sqrt{x} - \sqrt{y}| \leq |x - y|/2$.

For $f(x) = \sqrt{x}$, we have $f'(x) = \frac{1}{2\sqrt{x}}$. So the Mean Value Theorem shows that $\sqrt{x} - \sqrt{y} = \frac{1}{2\sqrt{c}}(x - y)$ for some c between x and y. Since $c > 1$, we have $\frac{1}{2\sqrt{c}} < \frac{1}{2}$, so $|\sqrt{x} - \sqrt{y}| = \left|\frac{1}{2\sqrt{c}}\right| \, |x - y| < \frac{1}{2}|x - y|$.

V. (8) For the function $f(x) = x^2$, write the Riemann sum of the form $\sum_{i=1}^{n} f(x_i)\Delta x$ for the partition of the interval $[0, 2]$ into n equal intervals, where $x_i = 2i/n$. Use the formula $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to rewrite this Riemann sum as an expression with only one term, then take its limit to find the area under f on $[0, 2]$.

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n} = \sum_{i=1}^{n} i^2 \cdot \frac{8}{n^3} = \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{8}{n^2}\frac{(n+1)(2n+1)}{6} .$$

Taking the limit, we have

$$\lim_{n\to\infty} \frac{8}{n^2}\frac{(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{4}{3}\frac{n+1}{n}\frac{2n+1}{n} = \lim_{n\to\infty} \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = \frac{4}{3} \cdot 1 \cdot 2 = \frac{8}{3} .$$

VI. (3) Write an expression, involving an integral, for a function $f(x)$ such that $f(2) = 0$ and f is an antiderivative of $B_2(x)$ ($B_2(x)$ is a certain continuous function defined on the entire real line, called the *Bessel function of the second kind.*)

$$\int_2^x B_2(t)\, dt$$

VII. (16) Calculate the following.

(a) $\int_0^1 (\sqrt{x} + 1)^2\, dx$

$$\int_0^1 (\sqrt{x} + 1)^2\, dx = \int_0^1 x + 2\sqrt{x} + 1\, dx = \frac{x^2}{2} + 2 \cdot \frac{2}{3} \cdot x^{3/2} + x\Big|_0^1 = \frac{1}{2} + \frac{4}{3} + 1 \left(= \frac{17}{6}\right).$$

(b) $\int_0^{3\pi/2} \sin(y)\, dy$

$$\int_0^{3\pi/2} \sin(y)\, dy = -\cos(y)\Big|_0^{3\pi/2} = -\cos(3\pi/2) - (-\cos(0)) = 0 - (-1) = 1.$$

(c) $\lim_{\text{mesh}(\{x_0,x_1,x_2,\ldots,x_n\})\to 0} \sum_{i=1}^{n} \sin(x_i^*)\, \Delta x_i$, where $\text{mesh}(\{x_0, x_1, x_2, \ldots, x_n\})$ is the mesh of the partition $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 3\pi/2$, $\Delta x_i = x_i - x_{i-1}$, and the limit is taken over all partitions and all choices of the sample points $x_i^* \in [x_{i-1}, x_i]$.

It equals $\int_0^{3\pi/2} \sin(x)\, dx = 1$ (from part (b)).

(d) The number $g(1)$, where $\frac{dg}{dx} = \frac{d}{dx}\left(\frac{(x^2 + 1)(x - 1)}{(x^4 + 1)^3}\right)$ and $g(0) = 4$.

g and $\frac{(x^2 + 1)(x - 1)}{(x^4 + 1)^3}$ are both antiderivatives of the same function, so $g(x) = \frac{(x^2 + 1)(x - 1)}{(x^4 + 1)^3} + C$ for some constant C. When $x = 0$, this becomes $4 = -1 + C$, so $C = 5$ and $g(x) = \frac{(x^2 + 1)(x - 1)}{(x^4 + 1)^3} + 5$. For $x = 1$, this says $g(1) = 0 + 5 = 5$.