Mathematics 2423-001 Name (please print) ____________________

Final Exam

December 16, 2011

Instructions: Give concise answers, but clearly indicate your reasoning. It is *not* expected that you will be able to answer all the questions, just do whatever you can.

I. (3) Explain how we know that every continuous function has an antiderivative.

If f is a continuous function, then (for any choice of a in the domain of f) the Fundamental Theorem of Calculus tells us that the function F defined by $F(x) = \int_a^x f(t)\,dt$ has derivative equal to f.

II. (6) For each of the following, write the partial fraction decomposition with unknown coefficients in the numerators, but do not go on to solve for the coefficients.

1. $\dfrac{1}{(x^2+x)^2}$

$$\frac{1}{(x^2+x)^2} = \frac{1}{(x+1)^2x^2} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{B_1}{x} + \frac{B_2}{x^2}$$

2. $\dfrac{1}{(x-1)(x^2+1)(x^4-1)}$

$$\frac{1}{(x-1)(x^2+1)(x^4-1)} = \frac{1}{(x-1)(x^2-1)(x^2+1)^2} = \frac{1}{(x-1)^2(x+1)(x^2+1)^2}$$
$$= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1}{x+1} + \frac{C_1x+D_1}{x^2+1} + \frac{C_2x+D_2}{(x^2+1)^2}$$

III. (6) (a) Briefly explain the idea of Simpson's Rule. Feel free to make use of a meaningful picture.

We partition $[a,b]$ with an even number n of equally spaced points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Taking two intervals at a time, we look at the three points (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}) on the graph of $f(x)$, for each even value of i. There is a unique parabola passing through those points, and the area under it between $x = x_i$ and $x = x_{i+2}$ approximates the area under $y = f(x)$ between these same x-values. Adding up these areas for each such pair of subintervals gives the estimate in Simpson's Rule.

(b) Given the following fact:

If $P(x)$ is the parabola passing through the points (x_0, y_0), (x_1, y_1), (x_2, y_2) with $h = x_1 - x_0 = x_2 - x_1$, then $\displaystyle\int_{x_0}^{x_2} P(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$.

obtain the formula in Simpson's Rule.

$$\frac{h}{3}(y_0+4y_1+y_2) + \frac{h}{3}(y_2+4y_3+y_4) + \frac{h}{3}(y_4+4y_5+y_6) + \cdots + \frac{h}{3}(y_{n-2}+4y_{n-1}+y_n)$$
$$= \frac{h}{3}(y_0+4y_1+y_2+y_2+4y_3+y_4+y_4+4y_5+y_6+\cdots+y_{n-2}+4y_{n-1}+y_n)$$
$$= \frac{h}{3}(y_0+4y_1+2y_2+4y_3+2y_4+4y_5+2y_6+\cdots+2y_{n-2}+4y_{n-1}+y_n)$$

IV. Find each of the following.
(20)

(a) $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx$ (you will need the trig identities $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$).

Performing the inverse substitution $x = \tan(\theta)$, $dx = \sec^2(\theta)\,d\theta$, we obtain

$$\begin{aligned}\int \frac{1}{(1+x^2)^2}\,dx &= \int \frac{\sec^2(\theta)}{\sec^4(\theta)}\,d\theta = \int \cos^2(\theta)\,d\theta = \int \frac{1}{2} + \frac{1}{2}\cos(2\theta)\,d\theta \\ &= \frac{\theta}{2} + \frac{1}{4}\sin(2\theta) + C = \frac{\theta}{2} + \frac{1}{2}\sin(\theta)\cos(\theta) + C\ .\end{aligned}$$

Since $x = \tan(\theta)$, the angle θ appears in a right triangle with opposite leg x and adjacent leg 1, showing that $\sin(\theta) = x/\sqrt{1+x^2}$, $\cos(\theta) = 1/\sqrt{1+x^2}$. Consequently,

$$\int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} + C = \frac{1}{2}\arctan(x) + \frac{x}{2(1+x^2)} + C\ .$$

(b) $\displaystyle\int \sin(\ln(x))\,dx$ (start by using the inverse substitution $x = e^u$, then integrate by parts twice)

Putting $x = e^u$, $dx = e^u\,du$, we have $\sin(\ln(x))\,dx = \int \sin(u)e^u\,du$. Now integrate by parts twice:

$$\int \sin(u)e^u\,du = \sin(u)e^u - \int \cos(u)e^u\,du = \sin(u)e^u - \cos(u)e^u - \int \sin(u)e^u\,du\ .$$

Since $u = \ln(x)$, solving for $\int \sin(u)e^u\,du$ gives

$$\int \sin(u)e^u\,du = \sin(u)e^u/2 - \cos(u)e^u/2 + C = x\sin(\ln(x))/2 - x\cos(\ln(x))/2 + C\ .$$

(c) $\displaystyle\lim_{x\to 0^+} \sin(x)\ln(x)$

$\displaystyle\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc(x)\cot(x)} = \lim_{x\to 0^+} \frac{\frac{\sin(x)}{x}}{-\cot(x)}$. Since $\displaystyle\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$ and $\displaystyle\lim_{x\to 0^+} -\cot(x) = -\infty$, $\displaystyle\lim_{x\to 0^+} \frac{\frac{\sin(x)}{x}}{-\cot(x)} = 0$.

Alternatively, one can calculate
$\displaystyle\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\sin(x)}{x}\frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{\sin(x)}{x}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -\sin(x) = 0.$

(d) $f(x)$, if $\displaystyle\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t}f(t)\,dt$

Using the Fundamental Theorem of Calculus to take derivatives, we find that $f(x) = e^{2x} + 2xe^{2x} + e^{-x}f(x)$. Solving for $f(x)$ gives

$$\begin{aligned} f(x)(1-e^{-x}) &= e^{2x} + 2xe^{2x} \\ f(x) &= \frac{e^{2x} + 2xe^{2x}}{1-e^{-x}} \end{aligned}$$

(e) $\displaystyle\int \frac{1}{\sqrt{x^2+x}}\,dx$, given that $\displaystyle\int \frac{1}{\sqrt{u^2-a^2}}\,du = \ln|u+\sqrt{u^2-a^2}|+C$.

$$\int \frac{1}{\sqrt{x^2+x}}\,dx = \int \frac{1}{\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}}}\,dx = \int \frac{1}{\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}}\,dx$$

$$= \int \frac{1}{\sqrt{u^2-\frac{1}{4}}}\,du = \ln\left|u+\sqrt{u^2-\frac{1}{4}}\right|+C = \ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C$$

This integral can also be evaluated using a clever substitution that one student almost found: Put $x=\sinh^2(u)$, $dx = 2\sinh(u)\cosh(u)\,du$. We then have:

$$\int \frac{1}{\sqrt{x^2+x}}\,dx = \int \frac{2\sinh(u)\cosh(u)}{\sqrt{\sinh^4(u)+\sinh^2(u)}}\,du = \int \frac{2\sinh(u)\cosh(u)}{\sinh(u)\sqrt{\sinh^2(u)+1}}\,du$$

$$= \int \frac{2\cosh(u)}{\sqrt{\cosh^2(u)}}\,du = \int \frac{2\cosh(u)}{\cosh(u)}\,du = \int 2\,du = 2u+C = 2\sinh^{-1}(\sqrt{x})+C = 2\ln|\sqrt{x}+\sqrt{x+1}|+C$$

To reconcile this with the other answer, we have

$$2\ln|\sqrt{x}+\sqrt{x+1}|+C = \ln((\sqrt{x}+\sqrt{x+1})^2)+C = \ln(x+2(\sqrt{x}\sqrt{x+1})+x+1)+C$$

$$= \ln(2x+2(\sqrt{x^2+x})+1)+C = \ln(2(x+\sqrt{x^2+x}+\frac{1}{2}))+C$$

$$= \ln(x+\sqrt{x^2+x}+\frac{1}{2})+\ln(2)+C = \ln(x+\sqrt{x^2+x}+\frac{1}{2})+C$$

V. (9) This problem concerns the curve which is the portion of the graph $y = 3+\frac{1}{2}\cosh(2x)$ between $x=0$ and $x=1$.

(a) Find ds for this curve.

$$ds = \sqrt{1+(y')^2}\,dx = \sqrt{1+\sinh^2(2x)}\,dx = \sqrt{\cosh^2(2x)}\,dx = \cosh(2x)\,dx$$

(b) Calculate the length of the curve.

$$\int_0^1 \cosh(2x)\,dx = \frac{1}{2}\sinh(2x)\Big|_0^1 = \frac{1}{2}\sinh(2) = \frac{e^2-e^{-2}}{4}\ .$$

(c) Write an integral whose value equals the surface area produced when the curve is rotated about the x-axis, but do not evaluate the integral.

$$\int_0^1 2\pi(3+\frac{1}{2}\cosh(2x))\cosh(2x)\,dx$$

VI. (4) Carry out integration by parts to reduce the evaluation of $\displaystyle\int \frac{x\arctan(x)}{(1+x^2)^2}\,dx$ to a problem of integrating a rational function, but do not continue on to integrate that rational function.

Taking $u=\arctan(x)$, $du = \dfrac{1}{1+x^2}\,dx$, $v = -\dfrac{1}{2(1+x^2)}$, and $dv = \dfrac{x}{(1+x^2)^2}$, we have

$$\int \frac{x\arctan(x)}{(1+x^2)^2}\,dx = -\frac{\arctan(x)}{1+x^2} + \int \frac{1}{2(1+x^2)}\,dx$$

VII. (6) If $f(t)$ is continuous for $t \geq 0$, the *Laplace transform* of f is the function $F(s)$ defined by $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$. Find $F(s)$ if $f(t) = e^{kt}$. Be sure to tell the domain of this $F(s)$.

$$\begin{aligned} F(s) &= \int_0^\infty e^{kt}\, e^{-st}\, dt = \int_0^\infty e^{(k-s)t}\, dt = \lim_{b\to\infty} \int_0^b e^{(k-s)t}\, dt \\ &= \lim_{b\to\infty} \frac{1}{k-s} e^{(k-s)t}\Big|_0^b = \lim_{b\to\infty} \frac{1}{k-s} e^{(k-s)b} - \frac{1}{k-s} \end{aligned}$$

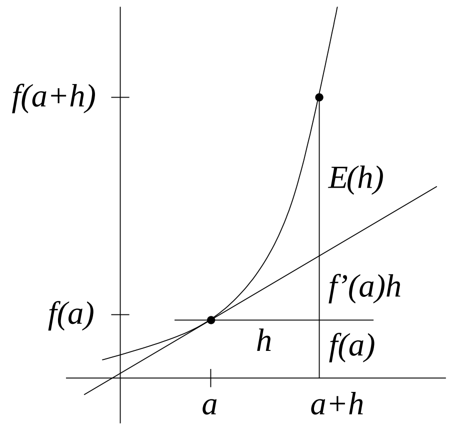
This is undefined when $k - s \geq 0$. For $k < s$, it equals $0 - \frac{1}{k-s} = \frac{1}{s-k}$. That is, $F(s) = \dfrac{1}{s-k}$ with domain the open interval (k, ∞).

VIII. (9) Recall that we defined $f'(a)$ by

$$f(a+h) = f(a) + f'(a)h + E(h)\ ,$$

where $\lim_{h\to 0} E(h)/h = 0$.

(a) Draw a picture showing the graph of a typical f, a, $a+h$, $f(a)$, $f(a+h)$, $f'(a)h$, and $E(h)$.



(b) Use the definition to find f' if $f(x) = x^2$.

$(a+h)^2 = a^2 + 2a \cdot h + h^2$. Since $\lim_{h\to 0} \dfrac{h^2}{h} = \lim_{h\to 0} h = 0$, $f'(a) = 2a$.

(c) Use integration by parts to show that $E(h) = \int_a^{a+h} (a+h-t) f''(t)\, dt$.

Integrating by parts with $u = a + h - t$, $du = -dt$, $v = f'(t)$, and $dv = f''(t)\, dt$, we have

$$\int_a^{a+h} (a+h-t) f''(t)\, dt = (a+h-t) f'(t)\Big|_a^{a+h} + \int_a^{a+h} f'(t)\, dt = -f'(a)h + f(a+h) - f(a) = E(h)\ .$$

IX. Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be a partition of the closed interval $[a, b]$.
(6)
(a) Define a *Riemann sum* for f on the interval $[a, b]$, associated to this partition.

It is a sum of the form $\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$ and each x_i^* lies in the subinterval $[x_{i-1}, x_i]$.

(b) Define $\int_a^b f(x)\,dx$

It is the limit of all Riemann sums of f associated to all partitions of $[a, b]$, where the limit is taken as the mesh of the partition (the largest of its Δx_i-values) limits to 0.

(c) For the function $f(x) = x^2$ and the interval $[-1, 2]$, find the smallest Riemann sum associated to the partition $-1 < -1/2 < 1 < 2$.

Take the x_i^* to be where the smallest values of x^2 occur in each of the intervals $[-1, -1/2]$, $[-1/2, 1]$, $[1, 2]$, that is, $-1/2$, 0, and 1 respectively, giving the sum $(-1/2)^2 \cdot 1/2 + 0^2 \cdot (3/2) + 1^2 \cdot 1 = 5/4$.

X. (a) State the Mean Value Theorem.
(6)

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists c between a and b such that $f(b) - f(a) = f'(c)(b - a)$.

(b) Let $F(x) = \int_0^x f(t)\,dt$. Tell why $\int_a^b f(t)\,dt = F(b) - F(a)$. Then use the Mean Value Theorem to tell why $\int_a^b f(t)\,dt = f(c)(b - a)$ for some c in the interval (a, b) (this is the "Mean Value Theorem for Integrals").

We have $F(b) - F(a) = \int_0^b f(t)\,dt - \int_0^a f(t)\,dt = \int_0^b f(t)\,dt + \int_a^0 f(t)\,dt = \int_a^b f(t)\,dt$. By the Fundamental Theorem of Calculus, $F'(x) = f(x)$, so using the Mean Value Theorem we have

$$\int_a^b f(t)\,dt = F(b) - F(a) = f(c)(b - a)$$

for some c between a and b.