Mathematics 1823-030
Examination III Form B
November 23, 2009

Name (please print) ______________________

Student Number ______________

(1) **Discussion Section** (circle day and time):

Th 9:00 Th 1:30 Th 3:00 F 8:30 F 9:30 F 2:30

I. (7) (a) Find the differential of $\sqrt{x}$.

$$d(\sqrt{x}) = \frac{d}{dx}(\sqrt{x})\,dx = \frac{1}{2\sqrt{x}}\,dx$$

(b) Use linear approximation to estimate $\sqrt{3.996}$.

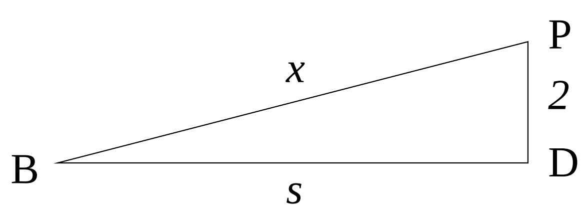
Using $f(a+\Delta x) \approx f(a) + f'(a)\Delta x$ with $f(x) = \sqrt{x}$, $a = 4$, and $\Delta x = -0.004$, we have

$$\sqrt{3.996} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(-0.002) = 2 + \frac{1}{4}(-0.004) = 1.999$$

(Alternatively, one can use $f(x+dx) \approx f(x) + dy$ with $f(x) = \sqrt{x}$, $x = 4$, and $dx = -0.004$, so that $dy = \frac{1}{4}dx$, to achieve the same estimate.)

II. (8) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 2 m higher than the bow of the boat. The rope is pulled at a rate of 2 m/sec.

(a) Draw and label a figure that illustrates this situation.



(b) How fast is the boat approaching the dock when it is 12 m from the dock?

$$x^2 = s^2 + 4$$
$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$
$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$

When $s = 12$, $x^2 = 144 + 4 = 148$, so $x = \sqrt{148}$.

$$\frac{dx}{dt} = -2, \text{ so}$$
$$\frac{ds}{dt} = \frac{\sqrt{148}}{12}(-2) = -\frac{\sqrt{148}}{6}$$

The boat is approaching the dock at $\frac{\sqrt{148}}{6}$ m/sec [which equals $\frac{\sqrt{37}}{3}$ m/sec].

III. (10) In this problem, $f(x) = \frac{2}{x} + \frac{1}{x^2}$, which can also be written as $\frac{2}{x^2}\big(x + \frac{1}{2}\big)$. The first and second derivatives of f are $f'(x) = -\frac{2}{x^2} - \frac{2}{x^3}$ and $f''(x) = \frac{4}{x^3} + \frac{6}{x^4}$ (do not check these)

1. Find the root or roots of f.

   $x = -\frac{1}{2}$

2. Find $\lim_{x\to\infty} f(x)$.

   $\lim_{x\to\infty} \frac{2}{x} + \frac{1}{x^2} = 0 + 0 = 0$

3. Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

   We think of these as $\lim_{x\to 0^+} \frac{2}{x^2}\big(x + \frac{1}{2}\big)$ and $\lim_{x\to 0^-} \frac{2}{x^2}\big(x + \frac{1}{2}\big)$. For x close to 0 and either positive or negative, $x + \frac{1}{2}$ is approximately $\frac{1}{2}$, while $\frac{2}{x^2}$ is large positive. So both limits are $+\infty$.

4. Find the critical number or critical numbers of f.

   $f'(x) = -\frac{2}{x^2} - \frac{2}{x^3} = -\frac{2}{x^3}(x+1)$, so $x = -1$ is the only critical number.

5. Find the inflection point or inflection points of f.

   $f''(x) = \frac{4}{x^3} + \frac{6}{x^4} = \frac{4}{x^4}(x + \frac{3}{2})$ so $x = -\frac{3}{2}$ is the only inflection point.

IV. (4) State the Mean Value Theorem, including its hypotheses.

If $f\colon [a,b] \to \mathbb{R}$ is a function which is

(i) continuous on $[a,b]$, and

(ii) differentiable on (a,b),

then there exists c between a and b such that $f(b) - f(a) = f'(c)(b-a)$.

V. (8) (a) For the function $\sqrt{x}$ on the interval $[0,9]$, find a number c that satisfies the conclusion of the Mean Value Theorem.

We have $\frac{f(9) - f(0)}{9 - 0} = \frac{1}{3}$ and $f'(x) = \frac{1}{2\sqrt{x}}$, so we seek c satisfying $f'(c) = \frac{1}{3}$. This is $\frac{1}{2\sqrt{c}} = \frac{1}{3}$, so $c = \frac{9}{4}$.

(b) Use the Mean Value Theorem to verify the following fact: Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) < 0$ for $a < x < b$, then $f(a) > f(b)$.

Applying the Mean Value Theorem, we have

$$f(b) - f(a) = f'(c)(b-a) \ .$$

We have $b - a > 0$, and $f'(c) < 0$ since $f'(x) < 0$ for all x between a and b. So $f'(c)(b-a) < 0$ and therefore $f(b) < f(a)$.

VI. A certain differentiable function f has domain all nonzero real numbers, and has the following properties:
(8)
(a) $f(-2) = 1$, $f(3) = 4$, $f(5) = 2$

(b) $f'(-2) = 0$, $f'(3) = 0$.

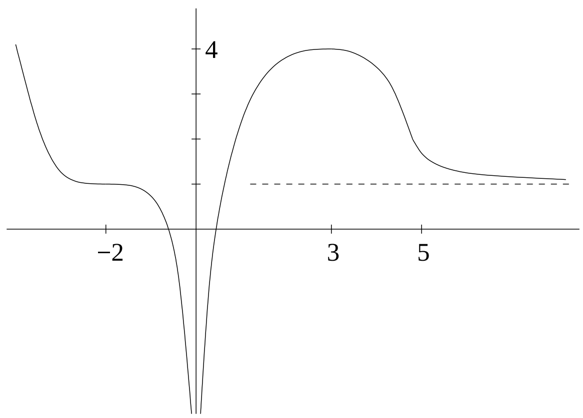
(c) $\lim_{x \to 0} f(x) = -\infty$.

(d) $f''(x) < 0$ for $-2 < x < 0$ and for $0 < x < 5$.

(e) $f''(x) > 0$ for $x < -2$ and $5 < x$.

(f) $\lim_{x \to \infty} f(x) = 1$.

Sketch a possible graph of f, using all of the above information.



VII. Let $f\colon [a,b] \to \mathbb{R}$ be a function which is continuous on $[a,b]$ and differentiable on (a,b). Label each of the
(10) following statements either T for true or F for false.

__T__ If a and b are roots of f, then there must exist a number c between a and b for which $f'(c) = 0$.

__T__ If $f'(x) < 0$ for $a < x < b$, then $f(a) < f(b)$.

__F__ The Mean Value Theorem is a special case of Rolle's Theorem.

__T__ The Mean Value Theorem can be deduced from Rolle's Theorem.

__T__ If $f(a) < f(b)$, then there must exist a number c between a and b for which $f'(c) > 0$.

__T or F__ If $f''(x)$ changes sign at c, then c is an inflection point of f. [The answer depends on the precise definition of inflection point that one uses.]

__T__ There must exist a number c in the interval $[a,b]$ such that $f(c) \geq f(x)$ for all x in $[a,b]$.

__F__ If f were not continuous on $[a,b]$, then it could still have an absolute maximum or absolute minimum value on $[a,b]$, but not both.

__F__ If c is an interior point of the interval and $f'(c) = 0$, then f must have either a local maximum or a local minimum (or both) at c.

__T__ If $f(x)$ equals the mass of the portion of a metal rod between 0 and x, then $f'(x)$ is the density function of the rod.