I. (a) Find the differential of $\sqrt{x}$.

\[ d(\sqrt{x}) = \frac{d}{dx}(\sqrt{x})\,dx = \frac{1}{2\sqrt{x}}\,dx \]

(b) Use linear approximation to estimate $\sqrt{3.996}$.

Using $f(a + \Delta x) \approx f(a) + f'(a)\Delta x$ with $f(x) = \sqrt{x}$, $a = 4$, and $\Delta x = -0.004$, we have

\[ \sqrt{3.996} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(-0.002) = 2 + \frac{1}{4}(-0.002) = 1.999 \]

(Alternatively, one can use $f(x + dx) \approx f(x) + dy$ with $f(x) = \sqrt{x}$, $x = 4$, and $dx = -0.004$, so that $dy = \frac{1}{4}dx$, to achieve the same estimate.)

II. (8) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 2 m higher than the bow of the boat. The rope is pulled at a rate of 2 m/sec.

(a) Draw and label a figure that illustrates this situation.

(b) How fast is the boat approaching the dock when it is 12 m from the dock?

\[ x^2 = s^2 + 4 \]

\[ 2x \frac{dx}{dt} = 2s \frac{ds}{dt} \]

\[ \frac{ds}{dt} = x \frac{dx}{dt} s \frac{dt}{dt} \]

When $s = 12$, $x^2 = 144 + 4 = 148$, so $x = \sqrt{148}$.

\[ \frac{dx}{dt} = -2, \text{ so} \]

\[ \frac{ds}{dt} = \frac{\sqrt{148}}{12}(-2) = -\frac{\sqrt{148}}{6} \]

The boat is approaching the dock at $\frac{\sqrt{148}}{6}$ m/sec [which equals $\frac{37}{3}$ m/sec].
III. In this problem, \( f(x) = \frac{2}{x^2} + \frac{1}{x^2} \), which can also be written as \( \frac{2}{x^2} (x + \frac{1}{2}) \). The first and second derivatives of \( f \) are \( f'(x) = -\frac{2}{x^3} - \frac{2}{x^4} \) and \( f''(x) = \frac{4}{x^3} + \frac{6}{x^4} \) (do not check these).

1. Find the root or roots of \( f \).
\[ x = -\frac{1}{2} \]

2. Find \( \lim_{x \to \infty} f(x) \).
\[ \lim_{x \to \infty} \frac{2}{x^2} + \frac{1}{x^2} = 0 + 0 = 0 \]

3. Find \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^-} f(x) \).

We think of these as \( \lim_{x \to 0^+} \frac{2}{x^2} \left(x + \frac{1}{2}\right) \) and \( \lim_{x \to 0^-} \frac{2}{x^2} \left(x + \frac{1}{2}\right) \). For \( x \) close to 0 and either positive or negative, \( x + \frac{1}{2} \) is approximately \( \frac{1}{2} \), while \( 2/x^2 \) is large positive. So both limits are \( +\infty \).

4. Find the critical number or critical numbers of \( f \).
\[ f'(x) = -\frac{2}{x^3} - \frac{2}{x^4} = -\frac{2}{x^4} (x + 1) \], so \( x = -1 \) is the only critical number.

5. Find the inflection point or inflection points of \( f \).
\[ f''(x) = \frac{4}{x^3} + \frac{6}{x^4} = \frac{4}{x^4} \left(x + \frac{3}{2}\right) \] so \( x = -\frac{3}{2} \) is the only inflection point.

IV. State the Mean Value Theorem, including its hypotheses.

(4) If \( f: [a, b] \to \mathbb{R} \) is a function which is
(i) continuous on \([a, b]\), and
(ii) differentiable on \((a, b)\),
then there exists \( c \) between \( a \) and \( b \) such that \( f(b) - f(a) = f'(c)(b - a) \).

V. (a) For the function \( \sqrt{x} \) on the interval \([0, 9]\), find a number \( c \) that satisfies the conclusion of the Mean Value Theorem.

We have \( \frac{f(9) - f(0)}{9 - 0} = \frac{1}{3} \) and \( f'(x) = \frac{1}{2\sqrt{x}} \), so we seek \( c \) satisfying \( f'(c) = \frac{1}{3} \). This is \( \frac{1}{2\sqrt{c}} = \frac{1}{3} \), so \( c = \frac{9}{4} \).

(b) Use the Mean Value Theorem to verify the following fact: Suppose that \( f: [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f'(x) < 0 \) for \( a < x < b \), then \( f(a) > f(b) \).

Applying the Mean Value Theorem, we have
\[ f(b) - f(a) = f'(c)(b - a) \]

We have \( b - a > 0 \), and \( f'(c) < 0 \) since \( f'(x) < 0 \) for all \( x \) between \( a \) and \( b \). So \( f'(c)(b - a) < 0 \) and therefore \( f(b) < f(a) \).
VI. A certain differentiable function $f$ has domain all nonzero real numbers, and has the following properties:

(a) $f(-2) = 1$, $f(3) = 4$, $f(5) = 2$

(b) $f'(-2) = 0$, $f'(3) = 0$.

(c) $\lim_{x \to 0} f(x) = -\infty$.

(d) $f''(x) < 0$ for $-2 < x < 0$ and for $0 < x < 5$.

(e) $f''(x) > 0$ for $x < -2$ and $5 < x$.

(f) $\lim_{x \to \infty} f(x) = 1$.

Sketch a possible graph of $f$, using all of the above information.

VII. Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on $(a, b)$. Label each of the following statements either $T$ for true or $F$ for false.

1. If $a$ and $b$ are roots of $f$, then there must exist a number $c$ between $a$ and $b$ for which $f'(c) = 0$.

2. If $f'(x) < 0$ for $a < x < b$, then $f(a) < f(b)$.

F The Mean Value Theorem is a special case of Rolle’s Theorem.

T The Mean Value Theorem can be deduced from Rolle’s Theorem.

T If $f(a) < f(b)$, then there must exist a number $c$ between $a$ and $b$ for which $f'(c) > 0$.

T or F If $f''(x)$ changes sign at $c$, then $c$ is an inflection point of $f$. [The answer depends on the precise definition of inflection point that one uses.]

T There must exist a number $c$ in the interval $[a, b]$ such that $f(c) \geq f(x)$ for all $x$ in $[a, b]$.

F If $f$ were not continuous on $[a, b]$, then it could still have an absolute maximum or absolute minimum value on $[a, b]$, but not both.

F If $c$ is an interior point of the interval and $f'(c) = 0$, then $f$ must have either a local maximum or a local minimum (or both) at $c$.

T If $f(x)$ equals the mass of the portion of a metal rod between 0 and $x$, then $f'(x)$ is the density function of the rod.