Mathematics 1823-030
Examination III Form A
November 23, 2009

Name (please print) ____________________

Student Number ____________________

(1) **Discussion Section** (circle day and time):

Th 9:00 Th 1:30 Th 3:00 F 8:30 F 9:30 F 2:30

I. (10) In this problem, $f(x) = \frac{2}{x} + \frac{1}{x^2}$, which can also be written as $\frac{2}{x^2}\left(x + \frac{1}{2}\right)$. The first and second derivatives of f are $f'(x) = -\frac{2}{x^2} - \frac{2}{x^3}$ and $f''(x) = \frac{4}{x^3} + \frac{6}{x^4}$ (do not check these)

1. Find the root or roots of f.

   $x = -\frac{1}{2}$

2. Find $\lim_{x\to\infty} f(x)$.

   $\lim_{x\to\infty} \frac{2}{x} + \frac{1}{x^2} = 0 + 0 = 0$

3. Find the critical number or critical numbers of f.

   $f'(x) = -\frac{2}{x^2} - \frac{2}{x^3} = -\frac{2}{x^3}(x+1)$, so $x = -1$ is the only critical number.

4. Find the inflection point or inflection points of f.

   $f''(x) = \frac{4}{x^3} + \frac{6}{x^4} = \frac{4}{x^4}(x + \frac{3}{2})$ so $x = -\frac{3}{2}$ is the only inflection point.

5. Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

   We think of these as $\lim_{x\to 0^+} \frac{2}{x^2}\left(x + \frac{1}{2}\right)$ and $\lim_{x\to 0^-} \frac{2}{x^2}\left(x + \frac{1}{2}\right)$. For x close to 0 and either positive or negative, $x + \frac{1}{2}$ is approximately $\frac{1}{2}$, while $\frac{2}{x^2}$ is large positive. So both limits are $+\infty$.

II. (7) (a) Find the differential of $\sqrt{x}$.

$$d(\sqrt{x}) = \frac{d}{dx}(\sqrt{x})\,dx = \frac{1}{2\sqrt{x}}\,dx$$

(b) Use linear approximation to estimate $\sqrt{8.994}$.

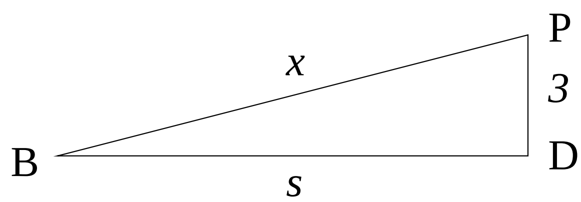
Using $f(a + \Delta x) \approx f(a) + f'(a)\Delta x$ with $f(x) = \sqrt{x}$, $a = 9$, and $\Delta x = -0.006$, we have

$$\sqrt{8.994} \approx \sqrt{9} + \frac{1}{2\sqrt{9}}(-0.006) = 3 + \frac{1}{6}(-0.006) = 2.999$$

(Alternatively, one can use $f(x + dx) \approx f(x) + dy$ with $f(x) = \sqrt{x}$, $x = 9$, and $dx = -0.006$, so that $dy = \frac{1}{6}dx$, to achieve the same estimate.)

III. (8) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 3 m higher than the bow of the boat. The rope is pulled at a rate of 3 m/sec.

(a) Draw and label a figure that illustrates this situation.



(b) How fast is the boat approaching the dock when it is 9 m from the dock?

$$x^2 = s^2 + 9$$
$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$
$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$
When $s = 9$, $x^2 = 81 + 9 = 90$, so $x = 3\sqrt{10}$.
$$\frac{dx}{dt} = -3, \text{ so}$$
$$\frac{ds}{dt} = \frac{3\sqrt{10}}{9}(-3) = -\sqrt{10}$$

The boat is approaching the dock at $\sqrt{10}$ m/sec.

IV. (8) (a) State the Mean Value Theorem, including its hypotheses.

If $f\colon [a,b] \to \mathbb{R}$ is a function which is

(i) continuous on $[a,b]$, and

(ii) differentiable on (a,b),

then there exists c between a and b such that $f(b) - f(a) = f'(c)(b-a)$.

(b) For the function $\sqrt{x}$ on the interval $[0,4]$, find a number c that satisfies the conclusion of the Mean Value Theorem.

We have $\dfrac{f(4)-f(0)}{4-0} = \dfrac{1}{2}$ and $f'(x) = \dfrac{1}{2\sqrt{x}}$, so we seek c satisfying $f'(c) = \dfrac{1}{2}$. This is $\dfrac{1}{2\sqrt{c}} = \dfrac{1}{2}$, so $c = 1$.

V. (4) Use the Mean Value Theorem to verify the following fact: Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) > 0$ for $a < x < b$, then $f(a) < f(b)$.

Applying the Mean Value Theorem, we have

$$f(b) - f(a) = f'(c)(b-a)\ .$$

We have $b - a > 0$, and $f'(c) > 0$ since $f'(x) > 0$ for all x between a and b. So $f'(c)(b-a) > 0$ and therefore $f(a) < f(b)$.

VI. (8) A certain differentiable function f has domain all real numbers except $x = -2$, and has the following properties:

(a) $f(-3) = -1$, $f(1) = 1$, $f(3) = 2$

(b) $f'(-3) = 0$, $f'(1) = 0$.

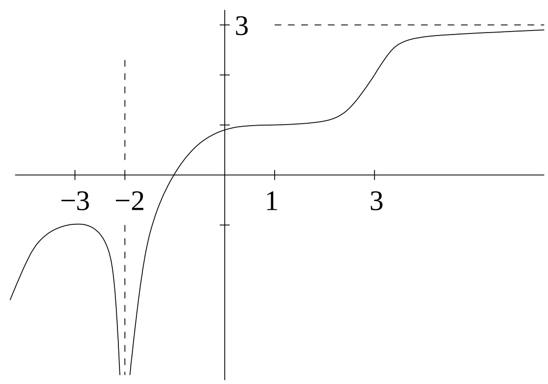
(c) $\lim_{x \to -2} f(x) = -\infty$.

(d) $f''(x) < 0$ for $x < -2$, for $-2 < x < 1$, and for $3 < x$.

(e) $f''(x) > 0$ for $1 < x < 3$.

(f) $\lim_{x \to \infty} f(x) = 3$.

Sketch a possible graph of f, using all of the above information.



VII. (10) Let $f\colon [a,b] \to \mathbb{R}$ be a function which is continuous on $[a,b]$ and differentiable on (a,b). Label each of the following statements either T for true or F for false.

F If $f'(x) < 0$ for $a < x < b$, then $f(a) < f(b)$.

T If a and b are roots of f, then there must exist a number c between a and b for which $f'(c) = 0$.

T The Mean Value Theorem can be deduced from Rolle's Theorem.

F The Mean Value Theorem is a special case of Rolle's Theorem.

T If $f(a) < f(b)$, then there must exist a number c between a and b for which $f'(c) > 0$.

T There must exist a number c in the interval $[a,b]$ such that $f(c) \geq f(x)$ for all x in $[a,b]$.

F If f were not continuous on $[a,b]$, then it could still have an absolute maximum or absolute minimum value on $[a,b]$, but not both.

F If c is an interior point of the interval and $f'(c) = 0$, then f must have either a local maximum or a local minimum (or both) at c.

T or F If $f''(x)$ changes sign at c, then c is an inflection point of f. [The answer depends on the precise definition of inflection point that one uses.]

T If $f(x)$ equals the mass of the portion of a metal rod between 0 and x, then $f'(x)$ is the density function of the rod.