Mathematics 1823-030
Examination II Form A
October 19, 2009

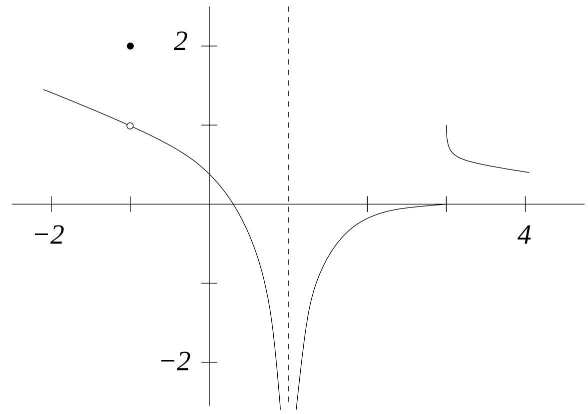
Name (please print) ____________________

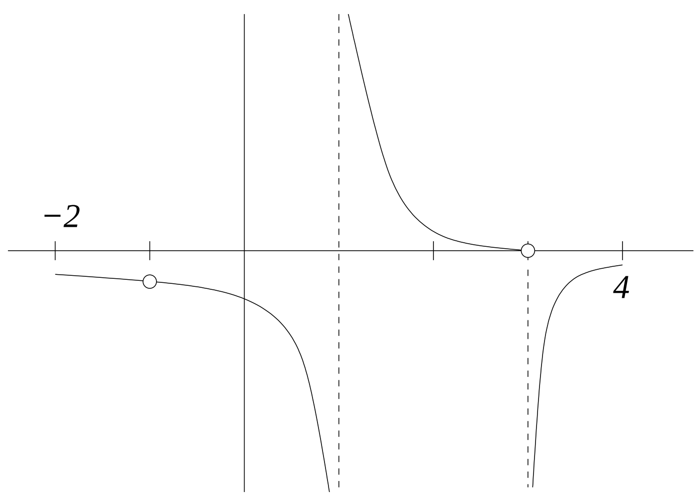
Student Number ____________________

(1) **Discussion Section** (circle day and time):

Th 9:00 Th 1:30 Th 3:00 F 8:30 F 9:30 F 2:30

I. (6) The figure to the right shows the graph of a certain function $f\colon [-2,4] \to \mathbb{R}$. On the coordinate system shown below, sketch a graph of the derivative $f'(x)$. The values of $f'(x)$ need not be precise, but should accurately reflect the behavior of $f(x)$. Indicate clearly any points where $f'(x)$ is underfined.





II. (4) Find the value of $\lim\limits_{\theta\to 0} \dfrac{\sin(5\theta)}{\sin(7\theta)}$ (not by plotting points or using l'Hôpital's rule).

$$\lim_{\theta\to 0} \frac{\sin(5\theta)}{\sin(7\theta)} = \lim_{\theta\to 0} \frac{\frac{\sin(5\theta)}{5\theta}\,5\theta}{\frac{\sin(7\theta)}{7\theta}\,7\theta} = \lim_{\theta\to 0} \frac{\frac{\sin(5\theta)}{5\theta}}{\frac{\sin(7\theta)}{7\theta}}\,\frac{5}{7}\,\frac{\theta}{\theta} = \frac{1}{1}\,\frac{5}{7}\,1 = \frac{5}{7}$$

III. (25) Calculate each of the following. When calculating derivatives, use the algebraic rules, the Chain Rule, and/or implicit differentiation (i. e. do not use the definition of the derivative as a limit). Do not perform simplifications to the answer unless instructed to do so.

(i) $\dfrac{dy}{dx}$ if $y = \dfrac{\sin(x^2)}{\sin^2(x)}$

$$\frac{d}{dx}\frac{\sin(x^2)}{\sin^2(x)} = \frac{\sin^2(x)\,\cos(x^2)\,2x - \sin(x^2)\,2\sin(x)\cos(x)}{\sin^4(x)} = \frac{2x\sin(x)\,\cos(x^2) - 2\cos(x)\sin(x^2)}{\sin^3(x)}$$

(final simplification not necessary)

(ii) $\dfrac{d^2y}{dx^2}$ if $y = \cot(x)$

$$\frac{d^2(\cot(x))}{dx^2} = \frac{d}{dx}(-\csc^2(x)) = -2\csc(x)\,(-\csc(x)\cot(x)) = 2\csc^2(x)\cot(x)$$

(iii) An equation for the tangent line to the curve $y = (x^2-1)^{1{,}000}$ at the point $(\sqrt{2}, 1)$. You need not simplify the answer.

$\dfrac{dy}{dx} = 1{,}000 \cdot (x^2-1)^{999} \cdot 2x$, so the slope of the tangent line is $\dfrac{dy}{dx}\Big|_{x=\sqrt{2}} = 1{,}000(1) \cdot 2\sqrt{2} = 2{,}000\sqrt{2}$.

Therefore an equation for the tangent line is

$$y - 1 = 2{,}000\sqrt{2}(x - \sqrt{2})$$
$$y = 2{,}000\sqrt{2}x - 3{,}999$$

(final simplification not necessary)

(iv) $\dfrac{dw}{dt}$ if $w = \sqrt{\dfrac{t-1}{t+1}}$. Simplify this answer.

$$\frac{d}{dt}\sqrt{\frac{t-1}{t+1}} = \frac{1}{2\sqrt{\dfrac{t-1}{t+1}}}\;\frac{(t+1)\cdot 1 - 1\cdot(t-1)}{(t+1)^2} = \frac{2}{2(t+1)^2\sqrt{\dfrac{t-1}{t+1}}} = -\frac{1}{(t+1)^{3/2}\sqrt{t-1}}$$

(v) $\dfrac{dy}{dx}$ if $x^2\cos(y) = y^3$

$$x^2\big(-\sin(y)\frac{dy}{dx}\big) + 2x\cos(y) = 3y^2\frac{dy}{dx}$$
$$(x^2\sin(y) + 3y^2)\frac{dy}{dx} = 2x\cos(y)$$
$$\frac{dy}{dx} = \frac{2x\cos(y)}{x^2\sin(y) + 3y^2}$$

IV. Define what it means to say that a function f is *continuous at* x_0. State the Intermediate Value Theorem.
(5)

f is *continuous* at x_0 when f is defined at x_0, $\lim_{x\to x_0} f(x)$ exists, and $\lim_{x\to x_0} f(x) = f(x_0)$.

The Intermediate Value Theorem says that if f is a continous function on the closed interval $[a,b]$, and N is any number between $f(a)$ and $f(b)$, then there exists a c between a and b such that $f(c) = N$.

V. State the precise, formal (i. e. using ϵ and δ) definition of: $\lim_{x\to 3\pi/4} \cot(x) = -1$.
(3)

For every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 3\pi/4| < \delta$, then $|\cot(x) + 1| < \epsilon$.

VI. Let f be a function which is differentiable at $x = a$. Label each of the following statements either T for true or F for false.
(6)

__T__ The limit $\lim_{w\to a} \frac{f(w) - f(a)}{w - a}$ must exist.

__F__ f must be differentiable on any open interval that contains a.

__T__ f must be continuous at $x = a$.

__F__ $f'(a)$ might be ∞ or $-\infty$.

VII. In the blank to the left of each of the following two questions, write the letter of the best response.
(4)

1. __B__ What type of mathematical object is $\frac{d^2y}{dx^2}$?

A) set B) function C) equation D) codomain E) number F) theorem

2. __J__ The pair $(4!, 0!)$ equals

A) $(4, 0)$ B) $(12, 0)$ C) $(20, 0)$ D) $(24, 0)$ E) $(32, 0)$ F) $(120, 0)$
G) $(4, 1)$ H) $(12, 1)$ I) $(20, 1)$ J) $(24, 1)$ K) $(32, 1)$ L) $(120, 1)$

VIII. The table to the right shows the values of the functions f, g, f', and g' at the x-values 1, 2, 3, and 4. For example, $f(4) = 2$ and $f'(4) = 4$. Write the value of each of the following:
(6)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	2	4	1
2	2	3	1	4
3	4	1	4	2
4	2	4	1	2

$(g \cdot f)'(3) =$ __12__ $(g \circ f)'(1) =$ __4__ $(f/g)'(3) =$ __$-\frac{1}{4}$__ $(f \circ f)(4) =$ __2__