I. The Law of Cosines formula is \( c^2 = a^2 + b^2 - 2ab \cos(\theta) \). Let \( T \) be a triangle with two sides of lengths \( 2\sqrt{2} \) meters and 3 meters, and suppose that the angle \( \theta \) between them is increasing at a rate of \( \sqrt{5} \) radians per second. Sketch and label the triangle, and use the Law of Cosines to find the rate of change of the length of the third side when \( \theta = \pi/4 \).

We have \( \frac{d\theta}{dt} = \sqrt{5} \) and we want to find \( \frac{dx}{dt} \) when \( \theta = \pi/4 \). Using the Law of Cosines and differentiating gives

\[
x^2 = (2\sqrt{2})^2 + 3^2 - 2 \cdot 2\sqrt{2} \cdot 3 \cdot \cos(\theta)
\]

\[
x^2 = 17 - 12\sqrt{2} \cos(\theta)
\]

\[
2x \frac{dx}{dt} = 12\sqrt{2} \sin(\theta) \frac{d\theta}{dt}
\]

\[
x \frac{dx}{dt} = 6\sqrt{2} \sin(\theta) \frac{d\theta}{dt}
\]

When \( \theta = \pi/4 \), we have \( x^2 = 17 - 12\sqrt{2}/\sqrt{2} = 5 \) so \( x = \sqrt{5} \), giving

\[
\sqrt{5} \frac{dx}{dt} = 6\sqrt{2}(1/\sqrt{2})(\sqrt{5}) = 6\sqrt{5}, \text{ so } \frac{dx}{dt} = 6
\]

The third side is increasing at a rate of 6 m/sec.

II. The graph below shows a function \( f \) having a root at \( x = 1 \).

(a) Suppose that Newton’s method is applied starting the iteration with \( x_1 = 2 \). On the graph below, indicate where \( x_2, x_3, \) and \( x_4 \) would be, drawing appropriate tangent lines to clarify the relation of each iterate with the previous one.

(b) Similarly, starting over with \( x_1 = 3 \) on the same graph below, indicate what \( x_2, x_3, \) and \( x_4 \) would be in that case.
III.  (a) Write the precise (i.e. epsilon-delta) definition of \( \lim_{x \to 2} 3x + 4 = 10 \).

For every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \), then \( |3x + 4 - 10| < \epsilon \).

(b) Use the precise definition to verify that \( \lim_{x \to 2} 3x + 4 = 10 \).

Let \( \epsilon > 0 \) be given. Put \( \delta = \epsilon/3 \). If \( 0 < |x - 2| < \delta \), then \( |3x + 4 - 10| = |3x - 6| = 3|x - 2| < 3\delta = \epsilon \).

IV.  The Mean Value Theorem implies that if \( g: \mathbb{R} \to \mathbb{R} \) is a differentiable function and \( g'(x) = 0 \) for all \( x \), then \( g \) is constant (do not verify this). Apply this fact to verify that if \( f: \mathbb{R} \to \mathbb{R} \) is a differentiable function \( F_1 \) and \( F_2 \) are antiderivatives of \( f \), then there exists some constant \( C \) such that \( F_1(x) = F_2(x) + C \) for all \( x \).

\[
\frac{d}{dx}(F_1(x) - F_2(x)) = F_1'(x) - F_2'(x) = 0 \quad \text{for all} \quad x.
\]
Therefore there exists a constant \( C \) so that \( F_1(x) - F_2(x) = C \). That is, \( F_1(x) = F_2(x) + C \).

V.  A farmer wants to fence an area of 600 square meters in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can this be done so as to minimize the cost of the fence?

The total length of fence is \( L(x) = 3x + 2 \frac{600}{x} = 3x + \frac{1200}{x} \), where \( x > 0 \). We have \( L'(x) = 3 - \frac{1200}{x^2} \), which has a critical number only when \( \frac{1200}{x^2} = 3 \), that is, when \( x = \sqrt{400} = 20 \). Since \( L''(x) = \frac{2400}{x^3} > 0 \), \( L(x) \) has an absolute minimum at \( x = 20 \). So the dimensions are 20 in the direction of the three fences, and \( \frac{600}{20} = 30 \) in the other direction.
VI. Calculate each of the following limits:

(6)

1. \( \lim_{x \to 0} \frac{\sin(5x)}{x} \)
   
   \[ \lim_{x \to 0} \frac{\sin(5x)}{x} = \lim_{x \to 0} 5 \frac{\sin(5x)}{5x} = 5 \cdot 1 = 5 \]

2. \( \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + x}} \)
   
   \[ \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + x}} = \lim_{x \to -\infty} \frac{x}{x \sqrt{1 + 1/x}} = \lim_{x \to -\infty} \frac{1}{\sqrt{1 + 1/x}} = \frac{1}{\sqrt{1 + 0}} = 1 \]

3. \( \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + x}} \)
   
   \[ \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + x}} = \lim_{x \to -\infty} \frac{-x}{-x \sqrt{1 + 1/x}} = \lim_{x \to -\infty} \frac{1}{-\sqrt{1 + 1/x}} = \frac{1}{-\sqrt{1 + 0}} = -1 \]

VII. A certain differentiable function \( f(x) \) satisfies \( f(1) = 3 \) and \( f'(x) > 4 \) for all \( x \). Use the Mean Value Theorem to verify that \( f(5) > 19 \).

By the Mean Value Theorem, \( f(5) - f(1) = f'(c)(5 - 1) = 4f'(c) \) for some \( c \) between 1 and 5. Since \( f'(c) > 4 \), this becomes \( f(5) - 3 > 4 \cdot 4 \), so \( f(5) > 19 \).

VIII. Calculate \( \frac{dy}{dx} \) if \( \sin(y) = \sqrt{x^2 + y^2} \).

(5)

Using implicit differentiation,

\[
\cos(y) \frac{dy}{dx} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2}} (2x + 2y \frac{dy}{dx})
\]

\[ \cos(y) \sqrt{x^2 + y^2} \frac{dy}{dx} = x + y \frac{dy}{dx} \]

\[ \left( \cos(y) \sqrt{x^2 + y^2} - y \right) \frac{dy}{dx} = x \]

\[ \frac{dy}{dx} = \frac{x}{\cos(y) \sqrt{x^2 + y^2} - y} \]
IX. Find each of the following:

1. The most general form for an antiderivative of \( \sec^2(x) - 8x^3 + \sqrt{x} \) on the interval \(-\frac{\pi}{2} < x < \frac{\pi}{2}\):

\[
\tan(x) - 8x^4/4 + \frac{x^{3/2}}{3/2} + C = \tan(x) - 2x^4 + 2x^{3/2}/3 + C
\]

2. The most general form for \( f(x) \) if \( f''(x) = 12x - 4 \).

The most general form for \( f'(x) \) is \( f'(x) = 12x^2/2 - 4x + C = 6x^2 - 4x + C \), so the most general form for \( f(x) \) is \( f(x) = 6x^3/3 - 4x^2/2 + Cx + C_1 = 2x^3 - 2x^2 + Cx + C_1 \), where \( C \) and \( C_1 \) can be any constants.

3. \( f(x) \), if \( f'(x) = 6x^2 - 2 \) and \( f(\pi) = 2\pi^3 \).

We have for some \( C \) that \( f(x) = 6x^3/3 - 2x + C = 2x^3 - 2x + C \). When \( x = \pi \), this is \( 2\pi^3 = 2\pi^3 - 2\pi + C \), so \( C = 2\pi \) and therefore \( f(x) = 2x^3 - 2x + 2\pi \).

4. The differential of the function \( \csc^2(2x) \).

The derivative of \( \csc^2(2x) \) is \( 2 \csc(2x) \cdot (-\csc(2x) \cot(2x)) \cdot 2 = -4 \csc^2(2x) \cot(2x) \), so \( d(\csc^2(2x)) = -4 \csc^2(2x) \cot(2x) \, dx \).

X. Let \( f : \mathbb{R} \to \mathbb{R} \) (i.e., let \( f \) be a function from the real numbers to the real numbers). In the blank to the left of each of the following questions, write the letter of the best response.

1. __F__ What type of mathematical object is the graph of \( f \)?
   - A) function  B) equation  C) codomain  D) number  E) operation  F) set

2. __D__ If it exists, what type of mathematical object is \( \lim_{x \to 2} f(x) \)?
   - A) function  B) equation  C) codomain  D) number  E) operation  F) set

3. __D__ If it exists, what type of mathematical object is \( \lim_{x \to \infty} f(x) \)?
   - A) function  B) equation  C) codomain  D) number  E) operation  F) set

4. __A__ If it exists, what type of mathematical object is an antiderivative of \( f \)?
   - A) function  B) equation  C) codomain  D) number  E) operation  F) set

5. __A__ What type of mathematical object is \( f \circ f \)?
   - A) function  B) equation  C) codomain  D) number  E) operation  F) set

6. __E__ What type of mathematical object is composition of functions?
   - A) function  B) equation  C) codomain  D) number  E) operation  F) set
XI. The table to the right shows the values of the functions $f$, $g$, $f'$, and $g'$ at the $x$-values 1, 2, 3, and 4. For example, $f(3) = 3$ and $f'(3) = 2$. Write the value of each of the following:

$$
\begin{array}{|c|c|c|c|c|}
\hline
x & f(x) & f'(x) & g(x) & g'(x) \\
\hline
1 & 2 & 4 & 1 & 2 \\
2 & 2 & 3 & 1 & 4 \\
3 & 3 & 2 & 4 & 1 \\
4 & 4 & 1 & 2 & 2 \\
\hline
\end{array}
$$

$$(g \cdot f)'(3) = 11 \quad (g \circ f)'(1) = 16 \quad (f/g)'(3) = \frac{5}{16} \quad (g \circ g)'(4) = 8$$

XII. Let $g : \mathbb{R} \to \mathbb{R}$ be a function, not necessarily differentiable or continuous. Label each of the following statements either $T$ for true or $F$ for false.

____ T If $g$ is continuous at $a$, then $\lim_{x \to a} g(x)$ must exist.

____ F If $\lim_{x \to a} g(x)$ exists, then $g$ must be continuous at $a$.

____ F If $f$ is continuous on a closed interval $[a, b]$, and $N$ is any value between $a$ and $b$, then there must exist $c$ between $f(a)$ and $f(b)$ such that $f(c) = N$.

____ F The Extreme Value Theorem applies only to differentiable functions.

____ F If $g$ is continuous at $a$, then it must be differentiable at $a$.

____ T If $g$ is differentiable at $a$, then it must be continuous at $a$.

____ T Every function of the form $\frac{1}{x} + C$ for some value of $C$ is an antiderivative of the function $-\frac{1}{x^2}$.

____ F Every antiderivative of the function $-\frac{1}{x^2}$ is of the form $\frac{1}{x} + C$ for some value of $C$.

____ F Both of the functions $\cos^2(x)$ and $\cos^2(x) + 5$ are antiderivatives of $-2 \sin(x)$.

____ T If $g$ is differentiable and $G$ is an antiderivative of $g$, then $g'$ tells the concavity of $G$.

____ F If $g$ is differentiable and $G$ is an antiderivative of $g$, then $g'$ tells where $G$ is increasing or decreasing.

____ T Newton’s method can be applied to find approximations to solutions to equations, as well as to find approximations to roots.