Mathematics 1823-030
Final Examination Form A
December 15, 2009

Name (please print) ______________________

Student Number ______________

I. (5) The Mean Value Theorem implies that if $g\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and $g'(x) = 0$ for all x, then g is constant (do not verify this). Apply this fact to verify that if $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and F_1 and F_2 are antiderivatives of f, then there exists some constant C such that $F_1(x) = F_2(x) + C$ for all x.

$\dfrac{d}{dx}\big(F_1(x) - F_2(x)\big) = F_1'(x) - F_2'(x) = 0$ for all x. Therefore there exists a constant C so that $F_1(x) - F_2(x) = C$. That is, $F_1(x) = F_2(x) + C$.

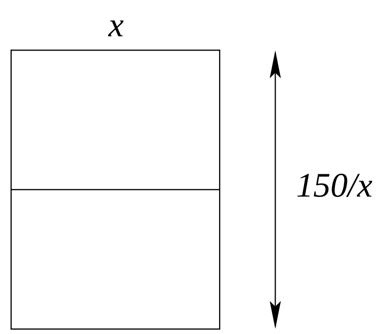
II. (7) (a) Write the precise (i. e. epsilon-delta) definition of $\lim_{x\to 3} 2x + 5 = 11$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|2x + 5 - 11| < \epsilon$.

(b) Use the precise definition to verify that $\lim_{x\to 3} 2x + 5 = 11$.

Let $\epsilon > 0$ be given. Put $\delta = \epsilon/2$. If $0 < |x - 3| < \delta$, then $|2x + 5 - 11| = |2x - 6| = 2|x - 3| < 2\delta = \epsilon$.

III. (7) A farmer wants to fence an area of 150 square meters in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can this be done so as to minimize the cost of the fence?

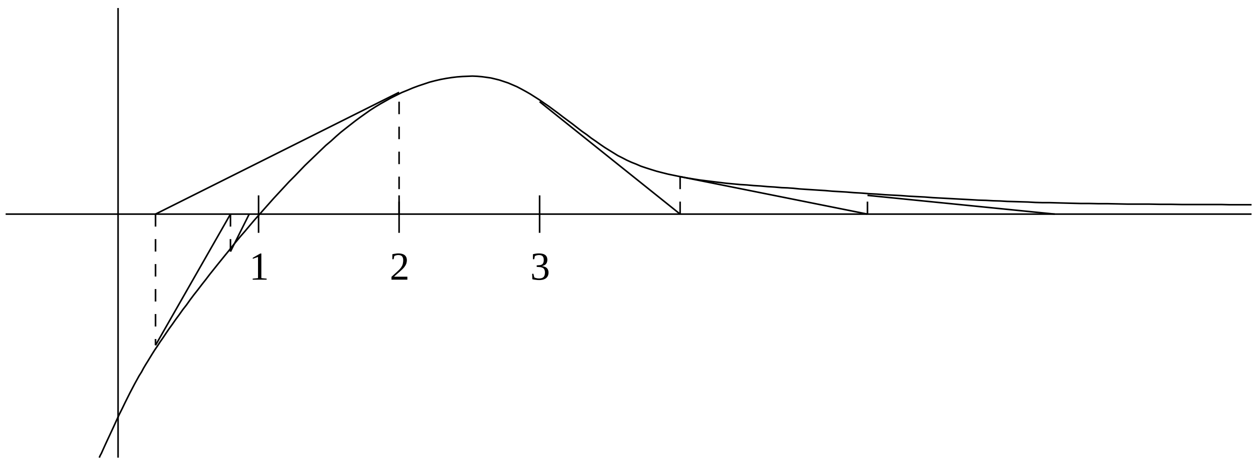


The total length of fence is $L(x) = 3x + 2\,\dfrac{150}{x} = 3x + \dfrac{300}{x}$, where $x > 0$. We have $L'(x) = 3 - \dfrac{300}{x^2}$, which has a critical number only when $\dfrac{300}{x^2} = 3$, that is, when $x = \sqrt{100} = 10$. Since $L''(x) = \dfrac{600}{x^3} > 0$, $L(x)$ has an absolute minimum at $x = 10$. So the dimensions are 10 in the direction of the three fences, and $\dfrac{150}{10} = 15$ in the other direction.

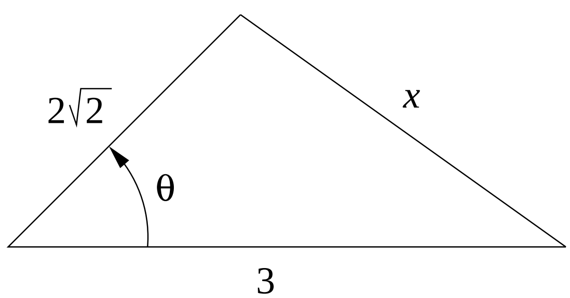
IV. (4) The graph below shows a function f having a root at $x = 1$.

(a) Suppose that Newton's method is applied starting the iteration with $x_1 = 2$. On the graph below, indicate where x_2, x_3, and x_4 would be, drawing appropriate tangent lines to clarify the relation of each iterate with the previous one.

(b) Similarly, starting over with $x_1 = 3$ on the same graph below, indicate what x_2, x_3, and x_4 would be in that case.



V. (6) The Law of Cosines formula is $c^2 = a^2 + b^2 - 2ab\cos(\theta)$. Let T be a triangle with two sides of lengths $2\sqrt{2}$ meters and 3 meters, and suppose that the angle θ between them is increasing at a rate of $\sqrt{5}$ radians per second. Sketch and label the triangle, and use the Law of Cosines to find the rate of change of the length of the third side when $\theta = \pi/4$.



We have $\dfrac{d\theta}{dt} = \sqrt{5}$ and we want to find $\dfrac{dx}{dt}$ when $\theta = \pi/4$. Using the Law of Cosines and differentiating gives

$$x^2 = (2\sqrt{2})^2 + 3^2 - 2\cdot 2\sqrt{2}\cdot 3\cdot\cos(\theta)$$
$$x^2 = 17 - 12\sqrt{2}\,\cos(\theta)$$
$$2x\frac{dx}{dt} = 12\sqrt{2}\,\sin(\theta)\frac{d\theta}{dt}$$
$$x\frac{dx}{dt} = 6\sqrt{2}\,\sin(\theta)\frac{d\theta}{dt}$$

When $\theta = \pi/4$, we have $x^2 = 17 - 12\sqrt{2}/\sqrt{2} = 5$ so $x = \sqrt{5}$, giving

$$\sqrt{5}\,\frac{dx}{dt} = 6\sqrt{2}(1/\sqrt{2})(\sqrt{5}) = 6\sqrt{5}, \text{ so } \frac{dx}{dt} = 6$$

The third side is increasing at a rate of 6 m/sec.

VI. (4) A certain differentiable function $f(x)$ satisfies $f(1) = 3$ and $f'(x) > 5$ for all x. Use the Mean Value Theorem to verify that $f(4) > 18$.

By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4-1) = 3f'(c)$ for some c between 1 and 4. Since $f'(c) > 5$, this becomes $f(4) - 3 > 3 \cdot 5$, so $f(4) > 18$.

VII. (5) Calculate $\dfrac{dy}{dx}$ if $\sqrt{x^2+y^2} = \sin(y)$.

Using implicit differentiation,

$$\begin{aligned} \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}\left(2x + 2y\frac{dy}{dx}\right) &= \cos(y)\frac{dy}{dx} \\ x + y\frac{dy}{dx} &= \cos(y)\sqrt{x^2+y^2}\frac{dy}{dx} \\ x &= \left(\cos(y)\sqrt{x^2+y^2} - y\right)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{x}{\cos(y)\sqrt{x^2+y^2} - y} \end{aligned}$$

VIII. (6) Calculate each of the following limits:

1. $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}}$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1+1/x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}} = \frac{1}{\sqrt{1+0}} = 1$$

2. $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+x}}$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+x}} = \lim_{x\to-\infty} \frac{x}{-x\sqrt{1+1/x}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x}} = \frac{1}{-\sqrt{1+0}} = -1$$

3. $\displaystyle\lim_{x\to0} \frac{\sin(7x)}{x}$

$$\lim_{x\to0} \frac{\sin(7x)}{x} = \lim_{x\to0} 7\,\frac{\sin(7x)}{7x} = 7 \cdot 1 = 7$$

IX. (12) Find each of the following:

1. The most general form for an antiderivative of $\csc^2(x) + 8x^3 - \sqrt{x}$ on the interval $0 < x < \pi$.

$$-\cot(x) + 8x^4/4 - \frac{x^{3/2}}{3/2} + C = -\cot(x) + 2x^4 - 2x^{3/2}/3 + C$$

2. $f(x)$, if $f'(x) = 6x^2 - 2$ and $f(\pi) = 2\pi^3$.

   We have for some C that $f(x) = 6x^3/3 - 2x + C = 2x^3 - 2x + C$. When $x = \pi$, this is $2\pi^3 = 2\pi^3 - 2\pi + C$, so $C = 2\pi$ and therefore $f(x) = 2x^3 - 2x + 2\pi$.

3. The most general form for $f(x)$ if $f''(x) = 12x - 4$.

   The most general form for $f'(x)$ is $f'(x) = 12x^2/2 - 4x + C = 6x^2 - 4x + C$, so the most general form for $f(x)$ is $f(x) = 6x^3/3 - 4x^2/2 + Cx + C_1 = 2x^3 - 2x^2 + Cx + C_1$, where C and C_1 can be any constants.

4. The differential of the function $\sec^2(2x)$.

   The derivative of $\sec^2(2x)$ is $2\sec(2x) \cdot \sec(2x)\tan(2x) \cdot 2 = 4\sec^2(2x)\tan(2x)$, so $d(\sec^2(2x)) = 4\sec^2(2x)\tan(2x)\,dx$.

X. (6) Let $f\colon \mathbb{R} \to \mathbb{R}$ (i. e. let f be a function from the real numbers to the real numbers). In the blank to the left of each of the following questions, write the letter of the best response.

1. __A__ What type of mathematical object is the graph of f?

   A) set B) function C) equation D) codomain E) number F) operation

2. __E__ If it exists, what type of mathematical object is $\lim_{x\to 2} f(x)$?

   A) set B) function C) equation D) codomain E) number F) operation

3. __E__ If it exists, what type of mathematical object is $\lim_{x\to\infty} f(x)$?

   A) set B) function C) equation D) codomain E) number F) operation

4. __B__ If it exists, what type of mathematical object is an antiderivative of f?

   A) set B) function C) equation D) codomain E) number F) operation

5. __B__ What type of mathematical object is $f \circ f$?

   A) set B) function C) equation D) codomain E) number F) operation

6. __F__ What type of mathematical object is composition of functions?

   A) set B) function C) equation D) codomain E) number F) operation

XI. (6) The table to the right shows the values of the functions f, g, f', and g' at the x-values 1, 2, 3, and 4. For example, $f(4) = 2$ and $f'(4) = 4$. Write the value of each of the following:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	3	1	4
2	3	2	4	1
3	4	1	4	2
4	2	4	1	2

$(g \cdot f)'(3) =$ __12__ $(g \circ f)'(1) =$ __3__ $(f/g)'(3) =$ __$-\frac{1}{4}$__ $(g \circ g)'(4) =$ __8__

XII. (12) Let $g\colon \mathbb{R} \to \mathbb{R}$ be a function, not necessarily differentiable or continuous. Label each of the following statements either T for true or F for false.

__F__ If $\lim_{x \to a} g(x)$ exists, then g must be continuous at a.

__T__ If g is continuous at a, then $\lim_{x \to a} g(x)$ must exist.

__F__ The Extreme Value Theorem applies only to differentiable functions.

__F__ If g is continuous at a, then it must be differentiable at a.

__T__ If g is differentiable at a, then it must be continuous at a.

__F__ If f is continuous on a closed interval $[a, b]$, and N is any value between a and b, then there must exist c between $f(a)$ and $f(b)$ such that $f(c) = N$.

__F__ Both of the functions $\cos^2(x)$ and $\cos^2(x) + 5$ are antiderivatives of $-2\sin(x)$.

__T__ Every function of the form $\frac{1}{x} + C$ for some value of C is an antiderivative of the function $-\frac{1}{x^2}$.

__F__ Every antiderivative of the function $-\frac{1}{x^2}$ is of the form $\frac{1}{x} + C$ for some value of C.

__T__ Newton's method can be applied to find approximations to solutions to equations, as well as to find approximations to roots.

__T__ If g is differentiable and G is an antiderivative of g, then the sign of g' tells the concavity of G.

__F__ If g is differentiable and G is an antiderivative of g, then the sign of g' tells where G is increasing or decreasing.