Mathematics 6813-001 Name (please print) ____________________

Final Examination

December 18, 2008

Instructions: Insofar as possible, give brief, clear answers. Use major theorems when possible.

I. (7) Let $p\colon (\widetilde{X},\widetilde{x}) \to (X,x)$ be a covering map, and let α be a loop in X based at x. Let $\widetilde{\alpha}$ be the lift of α starting at $\widetilde{x}$. Prove that $\widetilde{\alpha}$ is a loop if and only if $[\alpha] \in p_\#(\pi_1(\widetilde{X},\widetilde{x}))$.

Suppose first that $\widetilde{\alpha}$ is a loop. Then $[\widetilde{\alpha}] \in \pi_1(\widetilde{X},\widetilde{x})$, and $p_\#[\widetilde{\alpha}] = [p\circ\widetilde{\alpha}] = [\alpha]$, so $[\alpha] \in p_\#(\pi_1(\widetilde{X},\widetilde{x}))$.

Conversely, suppose that $[\alpha] \in p_\#(\pi_1(\widetilde{X},\widetilde{x}))$, say, $[\alpha] = p_\#[\beta] = [p\circ\beta]$. Choose a path homotopy from α to $p\circ\beta$. By the Homotopy Lifting Property, the homotopy lifts to a homotopy starting at $\widetilde{\alpha}$ and ending at a lift of $p\circ\beta$. Since it is a lift of a path homotopy, the lifted homotopy is a path homotopy as well (because at the endpoints, it is a lift of the constant path, which must be a constant path by uniqueness of lifts). The lifted homotopy ends at a lift of $p\circ\beta$ starting at $\widetilde{x}$, which by uniqueness of lifts must be β. Since β is a loop, so is $\widetilde{\alpha}$.

II. (7) Recall the proof that a path-connected, locally path-connected, semilocally simply connected space X has a simply-connected covering space $\widetilde{X}$. Tell how the points of $\widetilde{X}$ are defined, how the covering map $p\colon \widetilde{X}\to X$ is defined, and how the basic open sets in its topology are defined. You do not need to give any more details about the proof.

The elements of $\widetilde{X}$ are path homotopy classes of paths in X that start at the basepoint x_0 of X.

The covering map $p\colon \widetilde{X}\to X$ is defined by $p[\gamma] = \gamma(1)$.

The basic open sets are $U_{[\gamma]}$, where U is a path-connected open subset of X with $\pi_1(U)\to\pi_1(X)$ trivial at all basepoints, and γ is a path in X from x_0 to a point in U. The set $U_{[\gamma]}$ is then defined to be $\{[\gamma * \eta] \mid \eta \text{ is a path in } U \text{ starting at } \gamma(1)\}$.

III. (8) Define the following: Δ^n, a *singular n-simplex*, $C_n(X)$, $C_n(X,A)$, a *chain map*, $f_\#$, f_*. Show how the fact that $f_\#$ is a chain map proves that f_* is well-defined.

Δ^n is the convex hull of the standard basis vectors $\{e_1,\ldots,e_{n+1}\}$ in $\mathbb{R}^{n+1}$. Explicitly, a point in Δ^n can be written in barycentric coordinates as $\sum_{i=1}^{n+1} t_i e_i$ where each $0\le t_i\le 1$ and $\sum_{i=1}^{n+1} t_i = 1$. (One can also call the vertex set $[v_0,\ldots,v_n]$.)

A *singular n-simplex* is a continuous map $\sigma\colon \Delta^n \to X$.

$C_n(X)$ is the free abelian group on the set of all singular n-simplices.

$C_n(X,A)$ is the quotient group $C_n(X)/C_n(A)$.

A *chain map* between two chain complexes $\cdots \to A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \to \cdots$ and $\cdots \to B_{n+1} \xrightarrow{\partial} B_n \xrightarrow{\partial} B_{n-1} \to \cdots$ is a collection of homomorphisms $\psi\colon A_n \to B_n$ such that $\partial\psi = \psi\partial$.

For a continuous map $f\colon X\to Y$, $f_\#$ is the homomorphism from $C_n(X)$ to $C_n(Y)$ defined by $f_\#(\sum n_i\sigma_i) = \sum n_i\, f\circ\sigma_i$.

For a continuous map $f\colon X\to Y$, f_* is the homomorphism from $H_n(X)$ to $H_n(Y)$ defined by $f_*[z] = [f_\#(z)]$.

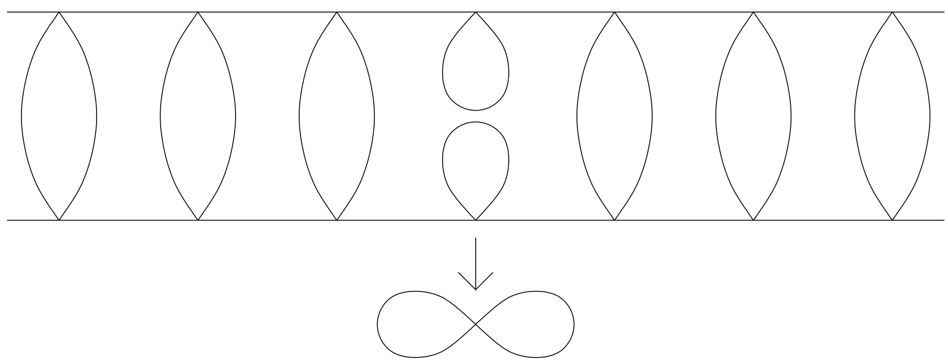
To prove that f_* is well-defined, suppose that $[z_1] = [z_2]$. Then $\partial z_1 = \partial z_2 = 0$, and $z_1 = z_2 + \partial c$ for some $(n+1)$-chain c. So $f_\#(z_1) = f_\#(z_2) + f_\#\partial(c) = f_\#(z_2) + \partial f_\#(c)$, so $[f_\#(z_1)] = [f_\#(z_2)]$ in $H_n(Y)$. [Alternatively, one can check that $f_\#$ takes cycles to cycles and boundaries to boundaries.]

IV. (6) State the *Homotopy Extension Property.* Use the fact that a subcomplex of a CW-complex has the HEP to prove the following proposition: Let A be a subcomplex of a CW-complex X. Suppose that $f\colon A \to Y$ is a continuous map that extends to a continuous map $F\colon X \to Y$. Suppose further that $f \simeq g$. Then g extends to a continuous map $G\colon X \to Y$.

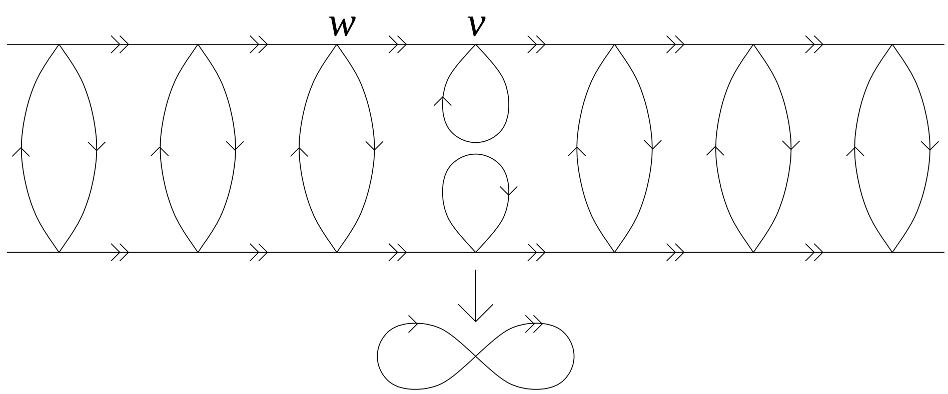
For $A \subset X$, one says that the pair (X, A) has the Homotopy Extension Property if whenever $f_t\colon A \to Y$ is a homotopy and $F_0\colon X \to Y$ satisfies $F_0|_A = f_0$, F_0 is the starting map of a homotopy $F_t\colon X \to Y$ such that for each t, $F_t|_A = f_t$.

To prove the proposition, we apply the Homotopy Extension Eroperty with f_t the homotopy from f to g and $F_0 = F$. The final map F_1 of the homotopy is then the desired extension G of g.

V. (6) The figure to the right shows a certain covering space of the one-point union of two circles a and b.



(i) Label a and b with single and double arrows. Make a corresponding labeling of the covering space that indicates a particular covering map.



(ii) Here is a sloppy way to state the Lifting Criterion: Let $p\colon \widetilde{X} \to X$ be a covering map, and let $f\colon Y \to X$ be a continuous map. Then f lifts to a map $F\colon Y \to \widetilde{X}$ with $pF = f$ if and only if $f_\#(\pi_1(Y)) \subseteq p_\#(\pi_1(\widetilde{X}))$. Give a precise statement of the Lifting Criterion, taking basepoints into account.

Let $p\colon (\widetilde{X}, \widetilde{x}) \to (X, x)$ be a covering map, and let $f\colon (Y, y) \to (X, x)$ be a basepoint-preserving continuous map. Then f lifts to a map $F\colon (Y, y) \to (\widetilde{X}, \widetilde{x})$ with $pF = f$ if and only if $f_\#(\pi_1(Y, y)) \subseteq p_\#(\pi_1(\widetilde{X}, \widetilde{x}))$.

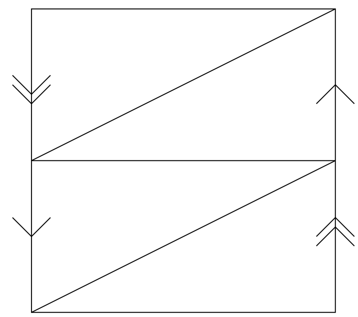
(iii) Use the example of a covering map given in part (i) to explain why basepoints must be taken into account in stating the Lifting Criterion.

Consider the inclusion of the left-hand circle into X. It lifts to a map to $\widetilde{X}$ taking its basepoint to v, but not to a map taking its basepoint to w, since there is no edge that forms a loop at w. So the existence of a lift depends on the basepoint used in $\widetilde{X}$.

VI. (6) Recall that the cone on a space A is the quotient space $CA = (A \times I)/(A \times \{1\})$. Let $A \subset X$, with A and X path-connected, and consider the quotient space $Y = X \cup CA$ obtained from X and CA by identifying each $(a, 0) \in CA$ with $a \in A \subset X$. Let P be the cone point $[A \times \{1\}]$. Observe that $CA - (A \times \{0\})$ is contractible, and $Y - P$ deformation retracts to X (you do not need to give any argument, except drawing reasonable pictures). Use van Kampen's Theorem to give a description of $\pi_1(Y, y_0)$ at a basepoint y_0 in $A \times (0, 1)$. (You can be a bit informal, but try to stay close to the statement of van Kampen's Theorem.)

Let $U = CA - (A \times \{0\})$ and $V = Y - P$. Since V deformation retracts to X, $\pi_1(V, y_0) \cong \pi_1(X, x_0)$ at some basepoint $x_0 \in X$. By van Kampen's Theorem (since $U \cap V = A \times (0, 1)$ is path-connected), the inclusions $i_\#\colon \pi_1(U, y_0) \to \pi_1(Y, y_0)$ and $j_\#\colon \pi_1(V, y_0) \to \pi_1(Y, y_0)$ induce a surjective homomorphism $\pi_1(U, y_0) * \pi_1(V, y_0) \to \pi_1(Y, y_0)$ whose kernel is the normal closure of the elements of the form $i_\#(\omega)j_\#(\omega^{-1})$ for $\omega \in \pi_1(U \cap V) \cong \pi_1(A)$. But U is contractible, so these are the elements $j_\#(\omega^{-1})$ for all $\omega \in \pi_1(U \times V)$. So the effect is to quotient out $\pi_1(X, x_0)$ by the normal closure of the image of $\pi_1(A, x_0)$ under the inclusion from A to X. That is, $\pi_1(X \cup CA) \cong \pi_1(X)/ \ll i_\#\pi_1(A) \gg$.

VII. (11) The figure to the right shows a Δ-structure on a Möbius band X; the right and left sides of the square are identified as indicated to form the band. The Δ-structure has four 2-simplices, seven 1-simplices, and three 0-simplices. The top and bottom horizontal 1-simplices t and b form the boundary circle in X. The middle horizontal 1-simplex m has its endpoints identified and forms the "core circle" C of X. Orient t, m, and b from left to right. It is easy to check that X deformation retracts to C (you do not need to prove this), so that the inclusion $i_*\colon H_k(C) \to H_k(X)$ is an isomorphism for each k.



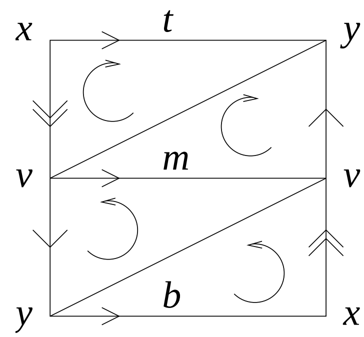
(i) The core circle C has a Δ-structure with one 1-simplex m and one 0-simplex v. Use this to calculate the homology of C. Since X deformation retracts to C, the inclusion $C \to X$ is an isomorphism on homology groups.

The Δ-chain complex $0 \to C_1(C) \xrightarrow{\partial_1} C_0(C) \to 0$ is $0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \to 0$, where the first summand is generated by m and the second by v. Since $\partial_1(m) = v - v = 0$, we have $H_1(C) = \ker(\partial_1)/\operatorname{im}(\partial_2) = \mathbb{Z}/\{0\} = \mathbb{Z}$ and $H_0(C) = \ker(\partial_0)/\operatorname{im}(\partial_1) = \mathbb{Z}/\{0\} = \mathbb{Z}$.

(ii) The boundary circle D of M has a Δ-structure with two 1-simplices t and b and two 0-simplices x and y, the left and right endpoints of t. Use this Δ-structure to calculate the homology of D.

The Δ-chain complex $0 \to C_1(C) \xrightarrow{\partial_1} C_0(C) \to 0$ is $0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \to 0$, where the summands of the first $\mathbb{Z} \oplus \mathbb{Z}$ are generated by t and b and those of the second by x and y. We have $\partial_1(t) = y - x$ and $\partial_1(b) = x - y$, so $\ker(\partial_1) \cong \mathbb{Z}$ generated by $t + b$, giving $H_1(C) = \ker(\partial_1)/\operatorname{im}(\partial_2) = \mathbb{Z} \oplus \mathbb{Z}/(1, 1) \cong \mathbb{Z}$ generated by $[t + b]$, and the image of δ_1 is generated by $x - y$ so $H_0(C) = \ker(\partial_0)/\operatorname{im}(\partial_1) = \mathbb{Z} \oplus \mathbb{Z}/\langle(1, -1) \cong \mathbb{Z}$.

(iii) Label orientations on the four 2-simplices τ_1, τ_2, τ_3, and τ_4 and on t, m, and b so that the 2-chain $c = \tau_1 + \tau_2 + \tau_3 + \tau_4$ has $\partial c = t + b - 2m$.



(iv) Use the chain in part (iii) (even if you did not find it explicitly) to explain why the inclusion $j\colon D \to X$ carries a generator of $H_1(D)$ to $2[m] \in H_1(X)$.

We have $j_*[t + b] = [t + b] = [t + b - \partial_2(\tau_1 + \tau_2 + \tau_3 + \tau_4)] = [t + b - t - b + 2m] = 2[m]$.

(v) Deduce that X does not retract to D.

Suppose that $r\colon X \to D$ is a retraction. We have ri equal to the identity on D, therefore r_*i_* is the identity on $H_1(D)$. But we have seen that $H_1(D) \xrightarrow{r_*} H_1(X) \xrightarrow{i_*} H_1(D)$ is $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$, where the first homomorphism is multiplication by 2. So 1=$r_*i_*(1) = r_*(2) = 2r_*(1)$, which is even. This is a contradiction.

VIII. (5) Let F and G be chain maps from the chain complex $\cdots \to A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \to \cdots$ to the chain complex $\cdots \to B_{n+1} \xrightarrow{\partial} B_n \xrightarrow{\partial} B_{n-1} \to \cdots$. Define a *chain homotopy* from F to G. Verify that if P is a chain homotopy from F to G, then $F_* = G_*\colon H_n(A) \to H_n(B)$.

A chain homotopy from F to G is a collection of homomorphisms $P\colon A_n \to B_{n+1}$ such that each $\partial P + P\partial = G - F$.

When a chain homotopy P exists, we have for any homology class $[z] \in H_n(A)$ that $G_*[z] - F_*[z] = [(G - F)(z)] = [\partial P(z) + P\partial(z)] = [\partial P(z)] = 0 \in H_n(B)$, using the facts that z is a cycle and the homology class of a boundary is 0. Therefore $F_*[z] = G_*[z]$.

IX. (8) Consider a commutative diagram of abelian groups and homomorphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow \\
0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \longrightarrow & 0
\end{array}$$

with exact rows.

(i) Prove that if α and γ are injective, then so is β.

$\ker(\beta) = 0$: Suppose that $\beta(b) = 0$. Then $0 = j'\beta(b) = \gamma(j(b))$. Since γ is injective, $j(b) = 0$ and by exactness there exists $a \in A$ with $i(a) = b$. We have $0 = \beta(b) = \beta(i(a)) = i'\alpha(a)$. Since i' is injective, this implies that $\alpha(a) = 0$, and since α is injective, $a = 0$. Therefore $b = i(a) = 0$.

(ii) Prove that if α and γ are surjective, then so is β.

β is surjective: Let $b' \in B'$. Since γ is surjective, there exists a $c \in C$ with $\gamma(c) = j'(b)$, and since j is surjective, there exists a $b \in B$ with $j(b) = c$. So $j'(\beta(b) - b') = \gamma j(b) - j'(b) = \gamma(c) - j'(b) = j'(b) - j'(b) = 0$. Therefore there is an $a' \in A$ with $i'(a') = \beta(b) - b'$. Since α is surjective, there exists $a \in A$ with $\alpha(a) = a'$. Then, $\beta(b - i(a)) = \beta(b) - \beta i(a) = \beta(b) - i'\alpha(a) = \beta(b) - i'(a') = \beta(b) - (\beta(b) - b') = b'$.

X. (11) Let X be the one-point union of two circles. For each of the following groups G, display a 4-fold covering space of X with deck transformation group G: $\{1\}$, C_2, $C_2 \times C_2$, C_4 (you do not need to verify that those are the deck transformation groups). Display an infinite-sheeted covering space of X with fundamental group $\mathbb{Z}$. Again, it is not necessary to verify that it is a covering, but use the single and double arrow method to clarify what the covering map is.