I. Let \( p: (\tilde{X}, \tilde{x}) \to (X, x) \) be a covering map, and let \( \alpha \) be a loop in \( X \) based at \( x \). Let \( \tilde{\alpha} \) be the lift of \( \alpha \) starting at \( \tilde{x} \). Prove that \( \tilde{\alpha} \) is a loop if and only if \( [\alpha] \in p_\#(\pi_1(\tilde{X}, \tilde{x})) \).

II. Recall the proof that a path-connected, locally path-connected, semilocally simply connected space \( X \) has a simply-connected covering space \( \tilde{X} \). Tell how the points of \( \tilde{X} \) are defined, how the covering map \( p: \tilde{X} \to X \) is defined, and how the basic open sets in its topology are defined. You do not need to give any more details about the proof.

III. Define the following: \( \Delta^n \), a singular \( n \)-simplex, \( C_n(X) \), \( C_n(X, A) \), a chain map, \( f_\# \), \( f_* \). Show how the fact that \( f_\# \) is a chain map proves that \( f_* \) is well-defined.

IV. State the Homotopy Extension Property. Use the fact that a subcomplex of a CW-complex has the HEP to prove the following proposition: Let \( A \) be a subcomplex of a CW-complex \( X \). Suppose that \( f: A \to Y \) is a continuous map that extends to a continuous map \( F: X \to Y \). Suppose further that \( f \simeq g \). Then \( g \) extends to a continuous map \( G: X \to Y \).

V. The figure to the right shows a certain covering space of the one-point union of two circles \( a \) and \( b \).

(i) Label \( a \) and \( b \) with single and double arrows. Make a corresponding labeling of the covering space that indicates a particular covering map.

(ii) Here is a sloppy way to state the Lifting Criterion: Let \( p: \tilde{X} \to X \) be a covering map, and let \( f: Y \to X \) be a continuous map. Then \( f \) lifts to a map \( F: Y \to \tilde{X} \) with \( pF = f \) if and only if \( f_\#(\pi_1(Y)) \subseteq p_\#(\pi_1(\tilde{X})) \). Give a precise statement of the Lifting Criterion, taking basepoints into account.

(iii) Use the example of a covering map given in part (i) to explain why basepoints must be taken into account in stating the Lifting Criterion.

VI. Recall that the cone on a space \( A \) is the quotient space \( CA = (A \times I)/(A \times \{1\}) \). Let \( A \subset X \), with \( A \) and \( X \) path-connected, and consider the quotient space \( Y = X \cup CA \) obtained from \( X \) and \( CA \) by identifying each \( (a, 0) \in CA \) with \( a \in A \subset X \). Let \( P \) be the cone point \( [A \times \{1\}] \). Observe that \( CA - (A \times \{0\}) \) is contractible, and \( Y - P \) deformation retracts to \( X \) (you do not need to give any argument, except drawing reasonable pictures). Use van Kampen’s Theorem to give a description of \( \pi_1(Y, y_0) \) at a basepoint \( y_0 \) in \( A \times (0,1) \). (You can be a bit informal, but try to stay close to the statement of van Kampen’s Theorem.)
VII. The figure to the right shows a $\Delta$-structure on a Möbius band $X$; the right and left sides of the square are identified as indicated to form the band. The $\Delta$-structure has four 2-simplices, seven 1-simplices, and three 0-simplices. The top and bottom horizontal 1-simplices $t$ and $b$ form the boundary circle in $X$. The middle horizontal 1-simplex $m$ has its endpoints identified and forms the “core circle” $C$ of $X$. Orient $t$, $m$, and $b$ from left to right. It is easy to check that $X$ deformation retracts to $C$ (you do not need to prove this), so that the inclusion $i_*$: $H_k(C) \to H_k(X)$ is an isomorphism for each $k$.

(i) The core circle $C$ has a $\Delta$-structure with one 1-simplex $m$ and one 0-simplex $v$. Use this to calculate the homology of $C$. Since $X$ deformation retracts to $C$, the inclusion $C \to X$ is an isomorphism on homology groups.

(ii) The boundary circle $D$ of $M$ has a $\Delta$-structure with two 1-simplices $t$ and $b$ and two 0-simplices $x$ and $y$, the left and right endpoints of $t$. Use this $\Delta$-structure to calculate the homology of $D$.

(iii) Label orientations on the four 2-simplices $\tau_1$, $\tau_2$, $\tau_3$, and $\tau_4$ and on $t$, $m$, and $b$ so that the 2-chain $c = \tau_1 + \tau_2 + \tau_3 + \tau_4$ has $\partial c = t + b - 2m$.

(iv) Use the chain in part (iii) (even if you did not find it explicitly) to explain why the inclusion $j$: $D \to X$ carries a generator of $H_1(D)$ to $2[m] \in H_1(X)$.

(v) Deduce that $X$ does not retract to $D$.

VIII. Let $F$ and $G$ be chain maps from the chain complex $\cdots \to A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \to \cdots$ to the chain complex $\cdots \to B_{n+1} \xrightarrow{\partial} B_n \xrightarrow{\partial} B_{n-1} \to \cdots$. Define a chain homotopy from $F$ to $G$. Verify that if $P$ is a chain homotopy from $F$ to $G$, then $F_*=G_*$: $H_n(A) \to H_n(B)$.

IX. Consider a commutative diagram of abelian groups and homomorphisms:

(i) Prove that if $\alpha$ and $\gamma$ are injective, then so is $\beta$.

(ii) Prove that if $\alpha$ and $\gamma$ are surjective, then so is $\beta$.

X. Let $X$ be the one-point union of two circles. For each of the following groups $G$, display a 4-fold covering space of $X$ with deck transformation group $G$: $\{1\}$, $C_2$, $C_2 \times C_2$, $C_4$ (you do not need to verify that those are the deck transformation groups). Display an infinite-sheeted covering space of $X$ with fundamental group $\mathbb{Z}$. Again, it is not necessary to verify that it is a covering, but use the single and double arrow method to clarify what the covering map is.