Mathematics 4513-001 Name (please print) ______________________

Examination I

October 16, 2008

Instructions: Give brief, clear answers. If asked for a definition, give the definition that we have used in this course. In some of the problems, you will need to calculate using the formula $\Omega_\ell X = X - 2\langle X - P, N\rangle N$.

I. (6) (a) Use the Orthonormal Basis Theorem to express the vector $(3,1)$ as a linear combination of the vectors in the orthonormal basis $\{(\frac{4}{5},\frac{3}{5}),(-\frac{3}{5},\frac{4}{5})\}$.

$$(3,1) = \langle (3,1),(\tfrac{4}{5},\tfrac{3}{5})\rangle(\tfrac{4}{5},\tfrac{3}{5}) + \langle (3,1),(-\tfrac{3}{5},\tfrac{4}{5})\rangle(-\tfrac{3}{5},\tfrac{4}{5}) = \tfrac{15}{5}(\tfrac{4}{5},\tfrac{3}{5}) - \tfrac{3}{5}(-\tfrac{3}{5},\tfrac{4}{5}) = 3(\tfrac{4}{5},\tfrac{3}{5}) - (-\tfrac{3}{5},\tfrac{4}{5}).$$

(b) Find an orthonormal basis for $\mathbb{R}^2$, one of whose vectors is proportional to the vector $(-2,3)$.

A unit vector in the direction of $(-2,3)$ is $(\frac{-2}{\sqrt{13}},\frac{3}{\sqrt{13}})$, and an orthonomal basis containing this vector is $\{(\frac{-2}{\sqrt{13}},\frac{3}{\sqrt{13}}),(\frac{-2}{\sqrt{13}},\frac{3}{\sqrt{13}})^\perp\} = \{(\frac{-2}{\sqrt{13}},\frac{3}{\sqrt{13}}),(\frac{-3}{\sqrt{13}},\frac{-2}{\sqrt{13}})\}$.

II. (5) The 3 Parallel Reflections Theorem says that if α, β, and γ are three lines perpendicular to a line ℓ, then there is a line δ perpendicular to ℓ so that $\Omega_\alpha\Omega_\beta\Omega_\gamma = \Omega_\delta$. Using this theorem, argue that if $F = \Omega_{\alpha_1}\Omega_{\alpha_2}\cdots\Omega_{\alpha_n}$ is a product of n reflections in lines perpendicular to ℓ, then F is either a translation (possibly the identity) or a reflection in a line perpendicular to ℓ.

If $n \geq 2$, then by the 3 Parallel Reflections Theorem, $\Omega_{\alpha_{n-2}}\Omega_{\alpha_{n-1}}\Omega_{\alpha_n} = \Omega_\delta$ for some line δ perpendicular to ℓ. Replacing $\Omega_{\alpha_{n-2}}\Omega_{\alpha_{n-1}}\Omega_{\alpha_n}$ by Ω_δ in the product $F = \Omega_{\alpha_1}\Omega_{\alpha_2}\cdots\Omega_{\alpha_n}$ gives expression for F as a product of only $n-2$ reflections. Since we can repeat this process as long as there are more than 2 reflections, we will eventually finish with either $F = \Omega_m$, in which case F is a reflection, or $\Omega_m\Omega_n$, in which case F is a translation (possibly the identity, when $m = n$) in the direction of ℓ.

III. (6) For a point $P \in \mathbb{R}^2$, define a function H_P from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $H_P X = 2P - X$.

(a) Verify that H_P is injective.

Suppose $H_P X = H_P Y$. Then $2P - X = 2P - Y$, so $-X = -Y$ and therefore $X = Y$.

(b) Verify that H_P^2 is the identity function of $\mathbb{R}^2$.

For all X, $H_P^2 X = H_P(H_P X) = H_P(2P - X) = 2P - (2P - X) = X$.

(c) Verify (algebraically) that $H_P H_Q = \tau_{2(P-Q)}$, where $\tau_v X = X + v$.

For all X, $H_P H_Q X = H_P(2Q - X) = 2P - (2Q - X) = X + 2(P - Q) = \tau_{2(P-Q)}X$.

IV. (6) Let $\ell = P + [v] = (3,2) + [(1,-2)]$.

(a) Find a unit normal N to ℓ.

A unit direction vector for ℓ is $(\frac{1}{\sqrt5},\frac{-2}{\sqrt5})$, so a unit normal is $N = (\frac{1}{\sqrt5},\frac{-2}{\sqrt5})^\perp = (\frac{2}{\sqrt5},\frac{1}{\sqrt5})$.

(b) By rewriting the equation $\langle X - P, N\rangle = 0$ in xy-coordinates, obtain an xy-equation for the line ℓ.

Writing $X = (x,y)$, we have
$0 = \langle (x,y) - (3,2),(\frac{2}{\sqrt5},\frac{1}{\sqrt5})\rangle = \langle (x-3,y-2),(\frac{2}{\sqrt5},\frac{1}{\sqrt5})\rangle = \frac{2}{\sqrt5}(x-3) + \frac{1}{\sqrt5}(y-2)$,
which may also be written as $2(x-3) + (y-2) = 0$ or $2x + y = 8$.

V. (6) (a) Define what it means to say that a function f is an *isometry* of $\mathbb{R}^2$.

It means that for all $X, Y \in \mathbb{R}^2$, $d(fX, fY) = d(x, y)$.

(b) Prove that if f and g are isometries of $\mathbb{R}^2$, then their composition fg is also an isometry.

For all $X, Y \in \mathbb{R}^2$, $d(fgX, fgY) = d(gX, gY) = d(X, Y)$, where the first equality uses the fact that f is an isometry, and the second uses the fact that g is an isometry.

(c) It is a fact that when $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry of $\mathbb{R}^2$, it has an inverse function $f^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ for which $ff^{-1} = id$ and $f^{-1}f = id$. Prove that if f is an isometry, then f^{-1} is also an isometry. Hint: Use the fact that $f(f^{-1}X) = X$.

For all $X, Y \in \mathbb{R}^2$, we have $d(X, Y) = d(f(f^{-1}X), f(f^{-1}Y)) = d(f^{-1}X, f^{-1}Y)$, where the last step uses the fact that f is an isometry.

VI. (5) Let $\mathrm{TR}(\ell)$ be the group of translations in the direction of ℓ. That is, if $\ell = P + [v]$, and τ_λ denotes the isometry of $\mathbb{R}^2$ given by $\tau_\lambda X = X + \lambda v$, then $\mathrm{TR}(\ell) = \{\tau_\lambda \mid \lambda \in \mathbb{R}\}$. Prove that the function $\Phi\colon \mathbb{R} \to \mathrm{TR}(\ell)$ defined by $\Phi(\lambda) = \tau_\lambda$ satisfies the homomorphism property $\Phi(\lambda_1 + \lambda_2) = \Phi(\lambda_1)\Phi(\lambda_2)$ (you do *not* need to show that Φ is injective or surjective).

For all X, we have

$$\begin{aligned}\Phi(\lambda_1 + \lambda_2)X &= \tau_{\lambda_1+\lambda_2}X = X + (\lambda_1 + \lambda_2)v = X + \lambda_1 v + \lambda_2 v \\ &= \tau_{\lambda_1}(X + \lambda_2 v) = \tau_{\lambda_1}\tau_{\lambda_2}X = \Phi(\lambda_1)\Phi(\lambda_2)X\end{aligned}$$

VII. (6) (a) Let H be a subgroup of a group G. Define a *coset* of H in G.

A coset of H in G is a subset of G of the form $Hg = \{hg \mid h \in H\}$.

(b) Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ be the group of integers, with the operation of addition, and let $4\mathbb{Z}$ be its subgroup $\{\ldots, -4, 0, 4, 8, \ldots\}$. Explain briefly how it is that $4\mathbb{Z} + 2 = 4\mathbb{Z} + 6$.

When we add 2 to each element of $4\mathbb{Z}$, we get

$$4\mathbb{Z} + 2 = \{\ldots, -8+2, -4+2, 0+2, 4+2, 8+2, \ldots\} = \{\ldots, -6, -2, 2, 6, 10, \ldots\}\ .$$

When we add 6 to each element of $4\mathbb{Z}$, we get

$$4\mathbb{Z} + 6 = \{\ldots, -8+6, -4+6, 0+6, 4+6, 8+6, \ldots\} = \{\ldots, -2, 2, 6, 10, 14, \ldots\}\ .$$

which equals $4\mathbb{Z} + 2$.

(c) List all the cosets of $4\mathbb{Z}$ in $\mathbb{Z}$.

The cosets are

$$\begin{aligned}4\mathbb{Z} + 0 &= \{\ldots, -4, 0, 4, 8, \ldots\} \\ 4\mathbb{Z} + 1 &= \{\ldots, -3, 1, 5, 9, \ldots\} \\ 4\mathbb{Z} + 2 &= \{\ldots, -2, 2, 6, 10, \ldots\} \\ 4\mathbb{Z} + 3 &= \{\ldots, -1, 3, 7, 11, \ldots\}\end{aligned}$$

(Once we get to $4\mathbb{Z} + 4 = \{\ldots, 0, 4, 8, 12, \ldots\} = 4\mathbb{Z}$, every coset equals one of these four, $4\mathbb{Z}$, $4\mathbb{Z} + 1$, $4\mathbb{Z} + 2$, or $4\mathbb{Z} + 3$. Also $4\mathbb{Z} + (-1) = 4\mathbb{Z} + 3$, $4\mathbb{Z} + (-2) = 4\mathbb{Z} + 2$, and so on for the cosets $4\mathbb{Z} + n$ with $n < 0$. So there are exactly these four cosets.)

VIII. Let P be a point in $\mathbb{R}^2$.
(6)

(a) Define what it means to say that an isometry R is a *rotation* about P.

It means that $R = \Omega_\alpha \Omega_\beta$ where α and β are two lines that contain P.

(b) Let α be a line passing through P. Let α_0 be the line through the origin 0 parallel to α, and let τ_P be the translation defined by $\tau_P X = X + P$. Verify by calculation that $\Omega_\alpha = \tau_P \Omega_{\alpha_0} \tau_{-P}$. Hint: Since α_0 passes through the origin, we have $\Omega_{\alpha_0} X = X - 2\langle X, N\rangle N$, where N is a unit normal to α_0 and α.

$$\tau_P \Omega_{\alpha_0} \tau_{-P} X = \tau_P \Omega_{\alpha_0}(X - P) = \tau_P(X - P - 2\langle X - P - 0, N\rangle N)$$
$$= X - P - 2\langle X - P - 0, N\rangle N + P = X - 2\langle X - P, N\rangle N = \Omega_\alpha X$$

IX. Use direct computation with the formula for $\Omega_\alpha X$ to show that if α_0 is a line through the origin, with unit
(6) normal vector N, then $\Omega_{\alpha_0}(X + Y) = \Omega_{\alpha_0}(X) + \Omega_{\alpha_0}(Y)$ for all X and Y in $\mathbb{R}^2$.

Taking $P = 0$ as our point on α_0, we have $\Omega_{\alpha_0} X = X - 2\langle X, N\rangle N$, so

$$\Omega_{\alpha_0}(X + Y) = X + Y - 2\langle X + Y, N\rangle N$$
$$= X + \langle X, N\rangle N + Y + \langle Y, N\rangle N = \Omega_{\alpha_0}(X) + \Omega_{\alpha_0}(Y) \ .$$

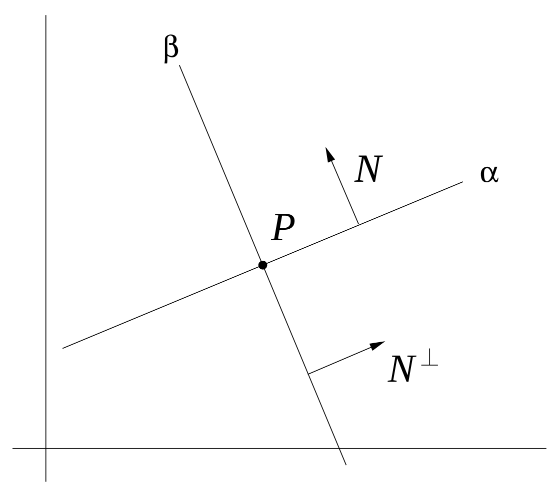
X. (a) Define what it means to say that an isometry J of $\mathbb{R}^2$ is a *glide-reflection.*
(5)

A glide-reflection is a reflection followed by a translation along its fixed line. (Alternatively, one can define it to be an isometry of the form $\tau_v \Omega_\ell$, where τ_v is a translation in the direction of ℓ.)

(b) Show that the composition of two glide reflections along *the same line* ℓ is a translation in the direction of ℓ (you may use the fact that Ω_ℓ commutes with any translation in the direction of ℓ).

Let $\tau_v \Omega_\ell$ and $\tau_w \Omega_\ell$ be two glide reflections along ℓ. Then $\tau_v \Omega_\ell \tau_w \Omega_\ell = \tau_v \tau_w \Omega_\ell \Omega_\ell = \tau_{v+w}$. Since v and w are both vectors in the direction of ℓ, so is $v + w$, so τ_{v+w} is a translation in the direction of ℓ.

XI. (Work on this one only if you are not short on time.) The
(6) figure to the right shows two perpendicular lines α and β that meet at the point P, and unit normal vectors N and $N^\perp$ to α and β. Calculate that $\Omega_\alpha \Omega_\beta X = 2P - X$ for all $X \in \mathbb{R}^2$.



We have for all X that

$$\begin{aligned}
\Omega_\alpha\Omega_\beta X &= \Omega_\alpha(X - 2\langle X - P, N^\perp\rangle N^\perp) \\
&= X - 2\langle X - P, N^\perp\rangle N^\perp - 2\langle X - 2\langle X - P, N^\perp\rangle N^\perp - P, N\rangle N \\
&= X - 2\langle X - P, N^\perp\rangle N^\perp - 2\langle X - P, N\rangle N - 2(\langle -2\langle X - P, N^\perp\rangle N^\perp, N\rangle N \\
&= X - 2\langle X - P, N^\perp\rangle N^\perp - 2\langle X - P, N\rangle N + 4\langle X - P, N^\perp\rangle\langle N^\perp, N\rangle N \\
&= X - 2\langle X - P, N^\perp\rangle N^\perp - 2\langle X - P, N\rangle N = X - 2(X - P) = 2P - X
\end{aligned}$$