(b) Find an orthonormal basis for \[ \langle 3, 1 \rangle. \]

A unit vector in the direction of \((-2, 3)\) is \( \left( \frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right) \), and an orthonormal basis containing this vector is \( \left\{ \left( \frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right), \left( \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \right) \right\}. \)

II. The 3 Parallel Reflections Theorem says that if \( \alpha, \beta, \) and \( \gamma \) are three lines perpendicular to a line \( \ell \), then there is a line \( \delta \) perpendicular to \( \ell \) so that \( \Omega_\alpha \Omega_\beta \Omega_\gamma = \Omega_\delta \). Using this theorem, argue that if \( F = \Omega_{\alpha_1} \Omega_{\alpha_2} \cdots \Omega_{\alpha_n} \) is a product of \( n \) reflections in lines perpendicular to \( \ell \), then \( F \) is either a translation (possibly the identity) or a reflection in a line perpendicular to \( \ell \).

If \( n \geq 2 \), then by the 3 Parallel Reflections Theorem, \( \Omega_{\alpha_{n-2}} \Omega_{\alpha_{n-1}} \Omega_{\alpha_n} = \Omega_\delta \) for some line \( \delta \) perpendicular to \( \ell \). Replacing \( \Omega_{\alpha_{n-2}} \Omega_{\alpha_{n-1}} \Omega_{\alpha_n} \) by \( \Omega_\delta \) in the product \( F = \Omega_{\alpha_1} \Omega_{\alpha_2} \cdots \Omega_{\alpha_n} \) gives expression for \( F \) as a product of only \( n-2 \) reflections. Since we can repeat this process as long as there are more than 2 reflections, we will eventually finish with either \( F = \Omega_m \), in which case \( F \) is a reflection, or \( \Omega_m \Omega_n \), in which case \( F \) is a translation (possibly the identity), when \( m = n \) in the direction of \( \ell \).

III. For a point \( P \in \mathbb{R}^2 \), define a function \( H_P \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) by \( H_P X = 2P - X \).

(a) Verify that \( H_P \) is injective.

Suppose \( H_P X = H_P Y \). Then \( 2P - X = 2P - Y \), so \( -X = -Y \) and therefore \( X = Y \).

(b) Verify that \( H_P^2 \) is the identity function of \( \mathbb{R}^2 \).

For all \( X \), \( H_P^2 X = H_P(H_P X) = H_P(2P - X) = 2P - (2P - X) = X. \)

(c) Verify (algebraically) that \( H_P H_Q = \tau_{2(P-Q)} \), where \( \tau_v X = X + v. \)

For all \( X \), \( H_P H_Q X = H_P(2Q - X) = 2P - (2Q - X) = X + 2(P - Q) = \tau_{2(P-Q)}X. \)

IV. Let \( \ell = P + [v] = (3, 2) + [(1, -2)]. \)

(a) Find a unit normal \( N \) to \( \ell \).

A unit direction vector for \( \ell \) is \( \left( \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}} \right) \), so a unit normal is \( N = \left( \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}} \right) = \left( \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right). \)

(b) By rewriting the equation \( \langle X - P, N \rangle = 0 \) in \( xy \)-coordinates, obtain an \( xy \)-equation for the line \( \ell \).

Writing \( X = (x, y) \), we have
\[
0 = \langle (x, y) - (3, 2), (\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \rangle = \langle (x-3, y-2), (\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \rangle = \frac{2}{\sqrt{3}}(x-3) + \frac{1}{\sqrt{3}}(y-2),
\]
which may also be written as \( 2(x-3) + (y-2) = 0 \) or \( 2x + y = 8 \).
V. (a) Define what it means to say that a function \( f \) is an isometry of \( \mathbb{R}^2 \).

It means that for all \( X, Y \in \mathbb{R}^2 \), \( d(fX, fY) = d(x, y) \).

(b) Prove that if \( f \) and \( g \) are isometries of \( \mathbb{R}^2 \), then their composition \( fg \) is also an isometry.

For all \( X, Y \in \mathbb{R}^2 \), \( d(fgX, fgY) = d(gX, gY) = d(X, Y) \), where the first equality uses the fact that \( f \) is an isometry, and the second uses the fact that \( g \) is an isometry.

(c) It is a fact that when \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is an isometry, it has an inverse function \( f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \) for which \( f f^{-1} = id \) and \( f^{-1} f = id \). Prove that if \( f \) is an isometry, then \( f^{-1} \) is also an isometry. Hint: Use the fact that \( f(f^{-1}X) = X \).

For all \( X, Y \in \mathbb{R}^2 \), we have \( d(X, Y) = d(f(f^{-1}X), f(f^{-1}Y)) = d(f^{-1}X, f^{-1}Y) \), where the last step uses the fact that \( f \) is an isometry.

VI. Let \( TR(\ell) \) be the group of translations in the direction of \( \ell \). That is, if \( \ell = P + [v] \), and \( \tau_\lambda \) denotes the isometry of \( \mathbb{R}^2 \) given by \( \tau_\lambda X = X + \lambda v \), then \( TR(\ell) = \{ \tau_\lambda \ | \ \lambda \in \mathbb{R} \} \). Prove that the function \( \Phi : \mathbb{R} \to TR(\ell) \) defined by \( \Phi(\lambda) = \tau_\lambda \) satisfies the homomorphism property \( \Phi(\lambda_1 + \lambda_2) = \Phi(\lambda_1)\Phi(\lambda_2) \) (you do not need to show that \( \Phi \) is injective or surjective).

For all \( X \), we have

\[
\Phi(\lambda_1 + \lambda_2)X = \tau_{\lambda_1+\lambda_2}X = X + (\lambda_1 + \lambda_2)v = X + \lambda_1 v + \lambda_2 v \\
= \tau_{\lambda_1}(X + \lambda_2 v) = \tau_{\lambda_1}\tau_{\lambda_2}X = \Phi(\lambda_1)\Phi(\lambda_2)X
\]

VII. (a) Let \( H \) be a subgroup of a group \( G \). Define a coset of \( H \) in \( G \).

A coset of \( H \) in \( G \) is a subset of \( G \) of the form \( Hg = \{ hg \ | h \in H \} \).

(b) Let \( \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, 3, \ldots \} \) be the group of integers, with the operation of addition, and let \( 4\mathbb{Z} \) be its subgroup \( \{ \ldots, -4, 0, 4, 8, \ldots \} \). Explain briefly how it is that \( 4\mathbb{Z} + 2 = 4\mathbb{Z} + 6 \).

When we add 2 to each element of \( 4\mathbb{Z} \), we get

\[
4\mathbb{Z} + 2 = \{ \ldots, -8 + 2, -4 + 2, 0 + 2, 4 + 2, 8 + 2, \ldots \} = \{ \ldots, -6, -2, 2, 6, 10, \ldots \}.
\]

When we add 6 to each element of \( 4\mathbb{Z} \), we get

\[
4\mathbb{Z} + 6 = \{ \ldots, -8 + 6, -4 + 6, 0 + 6, 4 + 6, 8 + 6, \ldots \} = \{ \ldots, -2, 2, 6, 10, 14, \ldots \}.
\]

which equals \( 4\mathbb{Z} + 2 \).

(c) List all the cosets of \( 4\mathbb{Z} \) in \( \mathbb{Z} \).

The cosets are

\[
4\mathbb{Z} + 0 = \{ \ldots, -4, 0, 4, 8, \ldots \}
\]

\[
4\mathbb{Z} + 1 = \{ \ldots, -3, 1, 5, 9, \ldots \}
\]

\[
4\mathbb{Z} + 2 = \{ \ldots, -2, 2, 6, 10, \ldots \}
\]

\[
4\mathbb{Z} + 3 = \{ \ldots, -1, 3, 7, 11, \ldots \}
\]

(Once we get to \( 4\mathbb{Z} + 4 = \{ \ldots, 0, 4, 8, 12, \ldots \} = 4\mathbb{Z} \), every coset equals one of these four, \( 4\mathbb{Z}, 4\mathbb{Z} + 1, 4\mathbb{Z} + 2, \) or \( 4\mathbb{Z} + 3 \). Also \( 4\mathbb{Z} + (-1) = 4\mathbb{Z} + 3, 4\mathbb{Z} + (-2) = 4\mathbb{Z} + 2, \) and so on for the cosets \( 4\mathbb{Z} + n \) with \( n < 0 \). So there are exactly these four cosets.)
VIII. Let $P$ be a point in $\mathbb{R}^2$.

(6) Define what it means to say that an isometry $R$ is a rotation about $P$.

It means that $R = \Omega_\alpha \Omega_\beta$ where $\alpha$ and $\beta$ are two lines that contain $P$.

(b) Let $\alpha$ be a line passing through $P$. Let $\alpha_0$ be the line through the origin 0 parallel to $\alpha$, and let $\tau_P$ be the translation defined by $\tau_P X = X + P$. Verify by calculation that $\Omega_\alpha = \tau_P \Omega_{\alpha_0} \tau_{-P}$. Hint: Since $\alpha_0$ passes through the origin, we have $\Omega_{\alpha_0} X = X - 2\langle X, N \rangle N$, where $N$ is a unit normal to $\alpha$.

\[
\tau_P \Omega_{\alpha_0} \tau_{-P} X = \tau_P \Omega_{\alpha_0} (X - P) = \tau_P (X - P - 2\langle X - P - 0, N \rangle N) = X - P - 2\langle X - P - 0, N \rangle N + P = X - 2\langle X - P, N \rangle N = \Omega_\alpha X
\]

IX. Use direct computation with the formula for $\Omega_\alpha X$ to show that if $\alpha_0$ is a line through the origin, with unit normal vector $N$, then $\Omega_{\alpha_0} (X + Y) = \Omega_{\alpha_0} (X) + \Omega_{\alpha_0} (Y)$ for all $X$ and $Y$ in $\mathbb{R}^2$.

Taking $P = 0$ as our point on $\alpha_0$, we have $\Omega_{\alpha_0} X = X - 2\langle X, N \rangle N$, so

\[
\Omega_{\alpha_0} (X + Y) = X + Y - 2\langle X + Y, N \rangle N = X + \langle X, N \rangle N + Y + \langle Y, N \rangle N = \Omega_{\alpha_0} (X) + \Omega_{\alpha_0} (Y).
\]

X. (a) Define what it means to say that an isometry $J$ of $\mathbb{R}^2$ is a glide-reflection.

A glide-reflection is a reflection followed by a translation along its fixed line. (Alternatively, one can define it to be an isometry of the form $\tau_v \Omega_\ell$, where $\tau_v$ is a translation in the direction of $\ell$.)

(b) Show that the composition of two glide reflections along the same line $\ell$ is a translation in the direction of $\ell$ (you may use the fact that $\Omega_\ell$ commutes with any translation in the direction of $\ell$).

Let $\tau_v \Omega_\ell$ and $\tau_w \Omega_\ell$ be two glide reflections along $\ell$. Then $\tau_v \Omega_\ell \tau_w \Omega_\ell = \tau_v \tau_w \Omega_\ell \Omega_\ell = \tau_{v + w}$. Since $v$ and $w$ are both vectors in the direction of $\ell$, so is $v + w$, so $\tau_{v + w}$ is a translation in the direction of $\ell$.

XI. (Work on this one only if you are not short on time.) The figure to the right shows two perpendicular lines $\alpha$ and $\beta$ that meet at the point $P$, and unit normal vectors $N$ and $N^\perp$ to $\alpha$ and $\beta$. Calculate that $\Omega_\alpha \Omega_\beta X = 2P - X$ for all $X \in \mathbb{R}^2$. 

We have for all $X$ that

$$\Omega_\alpha \Omega_\beta X = \Omega_\alpha (X - 2\langle X - P, N^\perp \rangle N^\perp)$$

$$= X - 2\langle X - P, N^\perp \rangle N^\perp - 2\langle X - 2(X - P, N^\perp)N^\perp - P, N \rangle N$$

$$= X - 2\langle X - P, N^\perp \rangle N^\perp - 2\langle X - P, N \rangle N - 2\langle -2(X - P, N^\perp)N^\perp, N \rangle N$$

$$= X - 2\langle X - P, N^\perp \rangle N^\perp - 2\langle X - P, N \rangle N + 4\langle X - P, N^\perp \rangle \langle N^\perp, N \rangle N$$

$$= X - 2\langle X - P, N^\perp \rangle N^\perp - 2\langle X - P, N \rangle N = X - 2(X - P) = 2P - X$$