Mathematics 4513-001 Name (please print) ______________________

Examination I

October 16, 2008

Instructions: Give brief, clear answers. If asked for a definition, give the definition that we have used in this course. In some of the problems, you will need to calculate using the formula $\Omega_\ell X = X - 2\langle X - P, N\rangle N$.

I. (6) (a) Use the Orthonormal Basis Theorem to express the vector $(3,1)$ as a linear combination of the vectors in the orthonormal basis $\{(\frac{4}{5},\frac{3}{5}),(-\frac{3}{5},\frac{4}{5})\}$.

(b) Find an orthonormal basis for $\mathbb{R}^2$, one of whose vectors is proportional to the vector $(-2,3)$.

II. (5) The 3 Parallel Reflections Theorem says that if α, β, and γ are three lines perpendicular to a line ℓ, then there is a line δ perpendicular to ℓ so that $\Omega_\alpha\Omega_\beta\Omega_\gamma = \Omega_\delta$. Using this theorem, argue that if $F = \Omega_{\alpha_1}\Omega_{\alpha_2}\cdots\Omega_{\alpha_n}$ is a product of n reflections in lines perpendicular to ℓ, then F is either a translation (possibly the identity) or a reflection in a line perpendicular to ℓ.

III. (6) For a point $P \in \mathbb{R}^2$, define a function H_P from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $H_P X = 2P - X$.

(a) Verify that H_P is injective.

(b) Verify that H_P^2 is the identity function of $\mathbb{R}^2$.

(c) Verify (algebraically) that $H_P H_Q = \tau_{2(P-Q)}$, where $\tau_v X = X + v$.

IV. (6) Let $\ell = P + [v] = (3,2) + [(1,-2)]$.

(a) Find a unit normal N to ℓ.

(b) By rewriting the equation $\langle X - P, N\rangle = 0$ in xy-coordinates, obtain an xy-equation for the line ℓ.

V. (6) (a) Define what it means to say that a function f is an *isometry* of $\mathbb{R}^2$.

(b) Prove that if f and g are isometries of $\mathbb{R}^2$, then their composition fg is also an isometry.

(c) It is a fact that when $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry of $\mathbb{R}^2$, it has an inverse function $f^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$ for which $ff^{-1} = id$ and $f^{-1}f = id$. Prove that if f is an isometry, then f^{-1} is also an isometry. Hint: Use the fact that $f(f^{-1}X) = X$.

VI. (5) Let $\mathrm{TR}(\ell)$ be the group of translations in the direction of ℓ. That is, if $\ell = P + [v]$, and τ_λ denotes the isometry of $\mathbb{R}^2$ given by $\tau_\lambda X = X + \lambda v$, then $\mathrm{TR}(\ell) = \{\tau_\lambda \mid \lambda \in \mathbb{R}\}$. Prove that the function $\Phi\colon \mathbb{R} \to \mathrm{TR}(\ell)$ defined by $\Phi(\lambda) = \tau_\lambda$ satisfies the homomorphism property $\Phi(\lambda_1 + \lambda_2) = \Phi(\lambda_1)\Phi(\lambda_2)$ (you do *not* need to show that Φ is injective or surjective).

VII. (6) (a) Let H be a subgroup of a group G. Define a *coset* of H in G.

(b) Let $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,3,\ldots\}$ be the group of integers, with the operation of addition, and let $4\mathbb{Z}$ be its subgroup $\{\ldots,-4,0,4,8,\ldots\}$. Explain briefly how it is that $4\mathbb{Z} + 2 = 4\mathbb{Z} + 6$.

(c) List all the cosets of $4\mathbb{Z}$ in $\mathbb{Z}$.

VIII. Let P be a point in $\mathbb{R}^2$.
(6)
(a) Define what it means to say that an isometry R is a *rotation* about P.

(b) Let α be a line passing through P. Let α_0 be the line through the origin 0 parallel to α, and let τ_P be the translation defined by $\tau_P X = X + P$. Verify by calculation that $\Omega_\alpha = \tau_P \Omega_{\alpha_0} \tau_{-P}$. Hint: Since α_0 passes through the origin, we have $\Omega_{\alpha_0} X = X - 2\langle X, N\rangle N$, where N is a unit normal to α_0 and α.

IX. Use direct computation with the formula for $\Omega_\alpha X$ to show that if α_0 is a line through the origin, with unit
(6) normal vector N, then $\Omega_{\alpha_0}(X + Y) = \Omega_{\alpha_0}(X) + \Omega_{\alpha_0}(Y)$ for all X and Y in $\mathbb{R}^2$.

X. (a) Define what it means to say that an isometry J of $\mathbb{R}^2$ is a *glide-reflection.*
(5)
(b) Show that the composition of two glide reflections along *the same line* ℓ is a translation in the direction of ℓ (you may use the fact that Ω_ℓ commutes with any translation in the direction of ℓ).

XI. (Work on this one only if you are not short on time.) The
(6) figure to the right shows two perpendicular lines α and β that meet at the point P, and unit normal vectors N and $N^\perp$ to α and β. Calculate that $\Omega_\alpha \Omega_\beta X = 2P - X$ for all $X \in \mathbb{R}^2$.