Mathematics 4513-001 Name (please print) ______________________

Final Examination

December 19, 2008

Instructions: Give brief, clear answers. If asked for a definition, give the definition that we have used in this course. In some of the problems, you will need to calculate using the formula $\Omega_\ell X = X - 2\langle X, P\rangle P$ for reflection of S^2 across the line with pole P.

I. (6) (i) Define a *line* in $\mathbb{R}^2$.

A *line* is a set of the form $P + [v]$, where P and v are vectors with $v \neq 0$, and $[v] = \{\lambda v \mid \lambda \in \mathbb{R}\}$.

(ii) Using the notation of the definition, write an expression for the line thorough $(1, 0)$ and $(3, 5)$.

For the line through $(1, 0)$ and $(3, 5)$, we can take $P = (1, 0)$ and $v = (3, 5) - (1, 0) = (2, 5)$, so the line can be written as $(1, 0) + [(2, 5)]$.

(iii) Find a unit normal to the line in (ii).

The direction vector is $(2, 5)$, so a normal vector is $(2, 5)^\perp = (-5, 2)$, and a unit normal is

$$\frac{(-5, 2)}{|(-5, 2)|} = \left(-\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right) .$$

II. (5) Let X be a set with a distance function $d\colon X \times X \to X$. Let f and g be functions from X to X.

(i) Define what it means to say that f is an *isometry* of X.

f is an isometry when $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

(ii) Prove that if f and g are isometries, then their composition fg is also an isometry.

For all $x, y \in X$, $d(fg(x), fg(y)) = d(g(x), g(y)) = d(x, y)$.

III. (7) Recall that for $a, b \in \mathbb{R}^3$ we defined the cross product $a \times b$ to be the unique vector in $\mathbb{R}^3$ such that $\langle a \times b, x\rangle = \det(x, a, b)$, where $\det(x, a, b)$ is the determinant of the matrix whose rows are x, a, and b. Use this definition to verify the following facts about the cross product. You may use standard properties of the determinant, and may use the fact that if $\langle a, x\rangle = \langle b, x\rangle$ for all x, then $a = b$.

(i) $a \times b = -b \times a$

For all x, $\langle a \times b, x\rangle = \det(x, a, b) = -\det(x, b, a) = -\langle b \times a, x\rangle = \langle -b \times a, x\rangle$, so $a \times b = -b \times a$.

(ii) $\langle a \times b, c\rangle = \langle a, b \times c\rangle$

$\langle a \times b, c\rangle = \det(c, a, b) = -\det(a, c, b) = \det(a, b, c) = \langle b \times c, a\rangle = \langle a, b \times c\rangle$.

(iii) If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, then $a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.

For all $x = (x_1, x_2, x_3)$,

$$\langle a \times b, x\rangle = \det \begin{bmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = x_1(a_2b_3 - a_3b_2) - x_2(a_1b_3 - a_3b_1) + x_3(a_1b_2 - a_2b_1)$$

$$= \langle (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1), (x_1, x_2, x_3)\rangle$$

for all x, so $a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.

IV. (6) Recall that a line in S^2 with pole P is defined to be the set $\{X \in S^2 \mid \langle X, P\rangle = 0\}$. Let P and Q be distinct points in S^2 with $P \neq -Q$.

(i) Give an expression (in terms of P and Q) for a pole of the line ℓ that contains P and Q.

The pole would be a unit vector perpendicular to P and Q, so we can take $P \times Q/|P \times Q|$.

(ii) Give an expression (in terms of P and Q) for a pole of the line perpendicular to ℓ that contains Q.

The pole would be perpendicular to the pole $P \times Q/|P \times Q|$ of ℓ, and hence perpendicular to $P \times Q$, and would also be perpendicular to Q since the line contains Q. So a pole would be $\dfrac{(P \times Q) \times Q}{|(P \times Q) \times Q|}$.

V. (9) Let G be the group whose elements are points in the plane $\mathbb{R}^2$, and whose operation is the usual vector addition, $(x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$. Define $\Phi \colon G \to G$ by $\Phi(X) = (3x, -2y)$, where $X = (x, y)$. Verify that Φ is an isomorphism (that is, verify that Φ is injective, surjective, and satisfies $\Phi(X_1 + X_2) = \Phi(X_1) + \Phi(X_2)$).

Injectivity: Suppose that $\Phi((x_1, y_1)) = \Phi((x_2, y_2))$. Then $(3x_1, -2y_1) = (3x_2, -2y_2)$. So $3x_1 = 3x_2$ and $-2y_1 = -2y_2$, giving $x_1 = x_2$ and $y_1 = y_2$. So $(x_1, y_1) = (x_2, y_2)$.

Surjectivity: Let $(x, y) \in G$. Then $\Psi(x/3, -y/2) = (x, y)$.

Homomorphism: $\Phi(X_1 + X_2) = \Phi((x_1, y_1) + (x_2, y_2)) = \Phi((x_1 + x_2, y_1 + y_2))$
$= (3(x_1 + x_2), -2(y_1 + y_2)) = (3x_1, -2y_1) + (3x_2, -2y_2) = \Phi(X_1) + \Phi(X_2)$.

VI. (5) (i) Calculate the determinants of the matrices $\mathrm{rot}(\theta)$ and $\mathrm{ref}(\theta)$.

$$\det(\mathrm{rot}(\theta)) = \det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \cos^2(\theta) - (-\sin^2(\theta)) = 1$$

$$\det(\mathrm{ref}(\theta)) = \det \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = -\cos^2(2\theta) - \sin^2(2\theta)) = -1$$

(ii) Explain geometrically why these are the values one would expect for these determinants.

Both of these linear transformations are isometries, so they preserve area, and therefore their determinants should have absolute value 1. Since $\mathrm{rot}(\theta)$ preserves the sense but $\mathrm{ref}(\theta)$ reverses it, we expect $\det(\mathrm{rot}(\theta)) = 1$ and $\det(\mathrm{ref}(\theta)) = -1$.

VII. (i) Give the definitions (in terms of reflections), of a translation and of a rotation of the *plane.*
(7)

A translation in the plane is a composition of two reflections that are perpendicular to a given line ℓ, that is, a composition of reflections in parallel lines. A rotation in the plane is a composition of two reflections in lines that meet at a point.

(ii) Explain why these concepts become the *same* concept when we are working on S^2.

In S^2, any two lines meet, so in some sense there is no direct analogue of a translation. But two lines perpendicular to a line ℓ meet in a pole of ℓ, so a product of two reflections in lines perpendicular to ℓ is a rotation about the pole of ℓ. In this sense, a translation (along ℓ) and a rotation (about a pole of ℓ) are the same thing.

VIII. (i) There is a version of the Three Reflections Theorem for the geometry of S^2. Tell what it says.
(6)

The Three Reflections Theorem in S^2 says that if α, β, and γ are lines meeting at a point P, then there is a line δ through P such that $\Omega_\alpha\Omega_\beta\Omega_\gamma = \Omega_\delta$.

(ii) Use the Three Reflections Theorem to show that if α, β, and ℓ are lines in S^2 that meet at a point P, then there is a line m through P so that $\Omega_\alpha\Omega_\beta = \Omega_\ell\Omega_m$.

By the Three Reflections Theorem, there is a line m through P so that $\Omega_\ell\Omega_\alpha\Omega_\beta = \Omega_m$. Composing with Ω_ℓ on the left, we have $\Omega_\alpha\Omega_\beta = \Omega_\ell^2\Omega_\alpha\Omega_\beta = \Omega_\ell\Omega_m$.

IX. Recall that we define the distance function in S^2 by $d(X,Y) = \cos^{-1}(\langle X,Y\rangle)$. Verify that J is an isometry
(5) of S^2 if and only if J preserves the inner product (that is, if and only if $\langle JX, JY\rangle = \langle X,Y\rangle$ for all X and Y in S^2).

We have

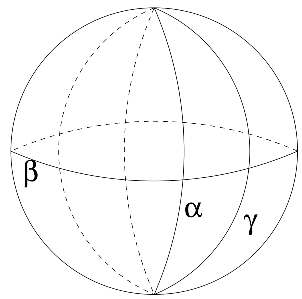
$$d(JX,JY) = d(X,Y) \iff \cos^{-1}(\langle JX,JY\rangle) = \cos^{-1}(\langle JX,JY\rangle) \iff \langle JX,JY\rangle = \langle X,Y\rangle\ ,$$

the latter equivalence because $\cos^{-1}$ is a bijective function.

X. Let α, β, and γ be lines in S^2 with α and γ both perpendicular to β.

(6)

(i) Sketch a picture of these three lines in S^2, oriented so that the intersection points of α and γ appear as the north and south poles.



(ii) Suppose that $\{e_1, e_2, e_3\}$ is an orthonormal basis for which e_3 is an intersection point of α and γ, say the north pole in part (i). Write the general form of the matrices of Ω_α, Ω_β, and Ω_γ with respect to this basis (you do not need to do any calculations, just use your knowledge of how the matrices of these reflections look for this kind of basis). It is a good idea to write them in block form, so that even though they are 3×3 matrices, they appear visually as 2×2 matrices.

$$\Omega_\alpha = \begin{bmatrix} \mathrm{ref}(\theta) & \\ & 1 \end{bmatrix}, \Omega_\beta = \begin{bmatrix} I & \\ & -1 \end{bmatrix}, \text{ and } \Omega_\gamma = \begin{bmatrix} \mathrm{ref}(\phi) & \\ & 1 \end{bmatrix}$$

where I is the 2×2 identity matrix.

(iii) By considering the product of the three matrices in part (ii), show that the composition $\Omega_\alpha \Omega_\beta \Omega_\gamma$ is a glide reflection.

Multiplying the three matrices gives

$$\begin{bmatrix} \mathrm{ref}(\theta)\,\mathrm{ref}(\phi) & \\ & -1 \end{bmatrix} = \begin{bmatrix} \mathrm{rot}(2(\theta - \phi)) & \\ & -1 \end{bmatrix}$$

which is the matrix of a glide reflection. Specifically, it is the product

$$\begin{bmatrix} \mathrm{rot}(2(\theta - \phi)) & \\ & 1 \end{bmatrix} \begin{bmatrix} I & \\ & -1 \end{bmatrix}$$

of a rotation about e_3 through the angle $2(\theta - \phi)$ and a reflection fixing e_1 and e_2 and sending e_3 to $-e_3$, i. e. the reflection Ω_β.

XI. Let $\{E_1, E_2, E_3\}$ be the standard basis of $\mathbb{R}^3$, and let J be an isometry of S^2. Use the Orthonormal Basis
(4) Theorem to express $J(X)$ in terms of E_1, E_2, and E_3.

$$JX = \langle JX, E_1\rangle E_1 + \langle JX, E_2\rangle E_2 + \langle JX, E_3\rangle E_3$$

A few people used the fact that isometries of S^2 are linear to give another response, which was also worth full credit:

$$JX = J(\langle X, E_1\rangle E_1 + \langle X, E_2\rangle E_2 + \langle X, E_3\rangle E_3) = \langle X, E_1\rangle JE_1 + \langle X, E_2\rangle JE_2 + \langle X, E_3\rangle JE_3$$

XII. (5) Let P, Q, and R be points in S^2, with $P \neq -Q$. Explain why P, Q, and R are collinear exactly when $\langle R, P \times Q \rangle = 0$.

A pole of the line containing P and Q is $\dfrac{P \times Q}{|P \times Q|}$. R lies in the line if and only if it is orthogonal to the pole, that is, if and only if $0 = \left\langle R, \dfrac{P \times Q}{|P \times Q|} \right\rangle = \dfrac{1}{|P \times Q|}\langle R, P \times Q \rangle$. Since $|P \times Q|$ is a nonzero scalar, the latter is equivalent to $\langle R, P \times Q \rangle = 0$.

XIII. (6) Let G be the group of isometries of S^2, and let H be the subgroup of G consisting of the isometries that take the point $(0,0,1)$ to itself, that is, the isometries J of S^2 with $J((0,0,1)) = (0,0,1)$.

(i) Tell what one would need to do to prove that H is not normal in G.

One would need to find an isometry J of S^2 with $J \in H$, and another isometry K of S^2 so that the composition KJK^{-1} is not in H. That is, $J(0,0,1) = (0,0,1)$ but $KJK^{-1}(0,0,1) \neq (0,0,1)$.

(ii) Prove that H is not normal in G. There are many ways to do this, here are three (the second and third are due to students in our class who did this problem, good work!):

Solution 1: Let ℓ be the equator, so ℓ has pole $(0,0,1)$, and $\Omega_\ell \in H$. Let e be the line of points equidistant from $(1,0,0)$ and $(0,0,1)$; a pole of e is

$$\frac{(1,0,0)-(0,0,1)}{|(1,0,0)-(0,0,1)|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) .$$

We know that Ω_e interchanges $(1,0,0)$ and $(0,0,1)$, and Ω_ℓ fixes every point on the equator ℓ, so we have

$$\Omega_e\Omega_\ell\Omega_e(0,0,1) = \Omega_e\Omega_\ell(1,0,0) = \Omega_e(1,0,0) = (0,0,1)$$

and therefore $\Omega_e\Omega_\ell\Omega_e \in H$. But

$$\Omega_e(\Omega_e\Omega_\ell\Omega_e)\Omega_e^{-1} = \Omega_e^2\Omega_\ell = \Omega_\ell ,$$

and $\Omega_\ell(0,0,1) = (0,0,-1)$ so $\Omega_\ell \notin H$. Therefore H is not a normal subgroup of G.

Solution 2: Let R be a rotation whose only fixed points are $(0,0,\pm1)$, and let m be any line such that $\Omega_m(0,0,1) \neq (0,0,\pm1)$ (that is, any line except the equator). Then $R \in H$. To show that $\Omega_m R \Omega_m^{-1} \notin H$, suppose for contradiction that that $\Omega_m R \Omega_m^{-1} \in H$. Using the fact that $\Omega_m = \Omega_m^{-1}$, we find

$$\begin{aligned} \Omega_m R \Omega_m^{-1}(0,0,1) &= (0,0,1) \\ \Omega_m^{-1} R \Omega_m(0,0,1) &= (0,0,1) \\ R(\Omega_m(0,0,1)) &= \Omega_m(0,0,1) \end{aligned}$$

Therefore $\Omega_m(0,0,1)$ is a fixed point of R, so $\Omega_m(0,0,1) = (0,0,\pm1)$, contradicting the choice of m. This contradiction shows that $\Omega_m R \Omega_m^{-1} \notin H$, so H is not a normal subgroup.

Solution 3: Let R be the rotation about $(1,0,0)$ that takes $(0,0,1)$ to $(0,1,0)$ (i. e. rotation by $\pi/2$). Note that $R^{-1}(0,0,1) = (0,-1,0)$. Let m be the line with pole $(0,1,0)$. Since $(0,0,1) \in m$, we have $\Omega_m(0,0,1) = (0,0,1)$ and hence $\Omega_m \in H$. But

$$R\Omega_m R^{-1}(0,0,1) = R\Omega_m(0,-1,0) = R(0,1,0) = (0,0,-1) ,$$

so $R\Omega_m R^{-1} \notin H$. Therefore H is not a normal subgroup.