\[x = \rho \cos(\theta) \sin(\phi), \quad y = \rho \sin(\theta) \sin(\phi), \quad z = \rho \cos(\phi),\]
\[dV = \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta, \quad \vec{r}_\phi \times \vec{r}_\theta = a \sin(\phi)(x\hat{i} + y\hat{j} + z\hat{k}),\]
\[dS = \sqrt{1 + g_x^2 + g_y^2} \, d\sigma\]
\[\int_S \vec{F} \cdot d\vec{S} = \int_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, d\sigma\]
\[\int_C (P \hat{i} + Q \hat{j} + R \hat{k}) \cdot d\vec{r} = \int_C P \, dx + Q \, dy + R \, dz\]

I. A path \(C\) is parameterized as a vector-valued function by \(\vec{r}(t) = t^2\hat{i} + t\hat{j}, 1 \leq t \leq 2\). Using this parameterization, evaluate the following line integrals.

1. \(\int_C (x/y) \, dx\)
2. \(\int_C (x/y) \, ds\)

II. Let \(\vec{F}(x, y, z) = 2xy\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + z)\hat{k}\).

1. Find a function \(f\) such that \(\vec{F} = \nabla f\).
2. Calculate \(\int_C \vec{F} \cdot d\vec{r}\), where \(C\) is given by the parameterization \(x = \cos^3(t), y = \cos^4(t), z = \sqrt{\cos(t)}, 0 \leq t \leq \pi/2\).

III. Let \(\vec{F}(x, y)\) be the vector field \(-y\hat{i} + x\hat{j} / (x^2 + y^2)^{3/2}\). Verify by calculation that \(\int_C \vec{F} \cdot d\vec{r}\) is not path-independent on the domain \(\{(x, y) \mid (x, y) \neq (0, 0)\}\). (Hint: Consider the line integral of \(\vec{F}\) on the unit circle \(C\)).

IV. Verify that if \(P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}\) is conservative, then \(\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}\). (Hint: if it is conservative, then it can be written in the form \(f_x \hat{i} + f_y \hat{j} + f_z \hat{k}\).)

V. Suppose that \(C\) is a closed loop with no self intersections, bounding a region \(D\).

1. Explain how one determines the “positive” or “standard” orientation on \(C\).
2. State Green’s Theorem.

VI. Calculate the curl and the divergence of the vector field \(\vec{F}(x, y, z) = 3z^2\hat{i} - x \cos(y)\hat{j} + 2xz\hat{k}\).

VII. Let \(S\) be the portion of the cylinder \(x^2 + z^2 = 1\) that lies between the vertical planes \(y = 0\) and \(y = 2 - x\).

1. Calculate \(\vec{r}_\theta\) and \(\vec{r}_h\).
2. Calculate \(\vec{r}_h \times \vec{r}_\theta\) and \(\|\vec{r}_h \times \vec{r}_\theta\|\).
VIII. Let $S$ be the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$. Calculate $dS$ in terms of $dD$, where $D$ is the domain in the $xy$-plane lying beneath $S$, and use it to calculate $\int \int_S z^2 \, dS$.

IX. Calculate $\int \int_S (xy \vec{i} + 4x^2 \vec{j} + yz \vec{k}) \cdot d\vec{S}$, where $S$ is the surface $z = xe^y$, $0 \leq x \leq 1$, $0 \leq y \leq 2$.

X. Use Green’s Theorem to calculate $\int_C (y^3 \vec{i} - x^3 \vec{j}) \cdot d\vec{r}$, where $C$ is the circle $x^2 + y^2 = 4$ with the clockwise orientation.