Mathematics 2443-003 Name (please print) ______________________

Examination III Form A

November 29, 2007

Instructions: Give brief answers, but clearly indicate your reasoning.

$x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$, $dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$, $\vec{r}_\phi \times \vec{r}_\theta = a\,\sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})$,
$\|\,\vec{r}_\phi \times \vec{r}_\theta\,\| = a^2\,\sin(\phi)$
$dS = \sqrt{1 + g_x^2 + g_y^2}\;dD$
$dS = \|\,\vec{r}_u \times \vec{r}_v\,\|\,dD$
$\iint_S\,\vec{F}\cdot d\vec{S} = \iint_S\,\vec{F}\cdot\vec{n}\,dS$
$\iint_S\,(P\,\vec{\imath} + Q\,\vec{\jmath} + R\,\vec{k})\cdot d\vec{S} = \iint_D\,-P\,g_x - Q\,g_y + R\,dD$
$\iint_S\,\vec{F}\cdot d\vec{S} = \iint_D\,\vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dD$

I. (6) A path C is parameterized as a vector-valued function by $\vec{r}(t) = t\vec{\imath} + t^2\vec{\jmath}$, $1 \le t \le 2$. Using this parameterization, evaluate the following line integrals.

1. $\displaystyle\int_C (y/x)\,dx$

   We have $dx = dt$, so $\displaystyle\int_C (y/x)\,dx = \int_1^2 (t^2/t)\,dt = \int_1^2 t\,dt = 3/2$.

2. $\displaystyle\int_C (y/x)\,ds$

   We have $ds^2 = dx^2 + dy^2 = (dt)^2 + (2t\,dt)^2 = (1 + 4t^2)\,dt^2$, so $ds = \sqrt{1+4t^2}\,dt$. So $\displaystyle\int_C (x/y)\,ds = \int_1^2 (t^2/t)\,\sqrt{1+4t^2}\;dt = \int_1^2 t\,\sqrt{1+4t^2}\;dt = (1/8)(2/3)(1+4t^2)^{3/2}\Big|_1^2 = (17\sqrt{17} - 5\sqrt{5})/12$.

II. (6) Let $\vec{F}(x,y,z) = 2xy\,\vec{\imath} + (x^2 + 2yz)\,\vec{\jmath} + (y^2 + 3z)\,\vec{k}$.

1. Find a function f such that $\vec{F} = \nabla f$.

   We need $f_x = 2xy$, so $f(x,y,z) = x^2y + g(y,z)$ for some function g. We also need $x^2 + 2yz = f_y = x^2 + g + y$, so $g_y = 2yz$ and therefore $g(y,z) = y^2z + h(z)$ and $f(x,y,z) = x^2y + y^2z + h(z)$. Finally, we need $y^2 + z = f_z = y^2 + h'(z)$, so $h'(z) = 3z$ and therefore $h(z) = 3z^2/2 + C$. So any f of the form $x^2y + y^2z + 3z^2/2 + C$ has $\nabla f = \vec{F}$.

2. Calculate $\displaystyle\int_C \vec{F}\cdot d\vec{r}$, where C is given by the parameterization $x = \sqrt{\cos(t)}$, $y = \cos^4(t)$, $z = \cos^5(t)$, $0 \le t \le \pi/2$.

   We apply the Fundamental Theorem for Line Integrals. The initial point of C is $(1,1,1)$, and its terminal point is $(0,0,0)$. So for the $f(x,y,z)$ in part 1 (taking $C = 0$) which had $\nabla f = \vec{F}$, we have $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(0,0,0) - f(1,1,1) = 0 - 7/2 = -7/2$.

III. (5) Let $\vec{F}(x,y)$ be the vector field $\dfrac{-y}{x^2+y^2}\vec{\imath}+\dfrac{x}{x^2+y^2}\vec{\jmath}$. Verify by calculation that $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ is not path-independent on the domain $\{(x,y) \mid (x,y)\neq(0,0)\}$. (Hint: Consider the line integral of $\vec{F}$ on the unit circle C).

On the unit circle, the unit tangent vector is $\vec{T}=-y\,\vec{\imath}+x\,\vec{\jmath}$, and $x^2+y^2=1$, so we have

$$\int_C \left(\frac{-y}{x^2+y^2}\vec{\imath}+\frac{x}{x^2+y^2}\vec{\jmath}\right)\cdot d\vec{r}=\int_C(-y\,\vec{\imath}+x\,\vec{\jmath})\cdot\vec{T}\,ds=\int_C y^2+x^2\,ds=\int_C 1\,ds=2\pi\ ,$$

since $\displaystyle\int_C 1\,ds$ is just the length of C. But when an integral is path-independent, the integral around any closed loop must be 0 (if the integral were path independent, then $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ this would be the same as the integral around the reverse path $-C$, which is -2π).

IV. (4) Verify that if $P(x,y,z)\,\vec{\imath}+Q(x,y,z)\,\vec{\jmath}+R(x,y,z)\,\vec{k}$ is conservative, then $\dfrac{\partial P}{\partial z}=\dfrac{\partial R}{\partial x}$. (Hint: if it is conservative, then it can be written in the form $f_x\,\vec{\imath}+f_y\,\vec{\jmath}+f_z\,\vec{k}$.)

A conservative vector field can be written in the form $f_x\,\vec{\imath}+f_y\,\vec{\jmath}+f_z\,\vec{k}$, that is, $P(x,y,z)=f_x$ and $R(x,y,z)=f_z$. So $\dfrac{\partial P}{\partial z}=f_{xz}$ and $\dfrac{\partial R}{\partial x}=f_{zx}$. By Clairaut's Theorem, these must be equal.

V. (5) Suppose that C is a closed loop with no self intersections, bounding a region D.

1. Explain how one determines the "positive" or "standard" orientation on C.

   If you travel along C in the positive direction, you see the region in the plane bounded by C on your left, rather than on your right.

2. State Green's Theorem.

   A closed loop C bounds a region R in the plane, and C is given the positive orientation. Green's Theorem says that for functions $P(x,y)$ and $Q(x,y)$,

   $$\int_C P(x,y)\,dx+Q(x,y)\,dy=\iint_R \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dR\ .$$

   (Alternatively, one can state this in terms of the line integral of a vector field:

   $$\int_C (P(x,y)\,\vec{\imath}+Q(x,y)\,\vec{\jmath})\cdot d\vec{r}=\iint_R \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dR\ .)$$

VI. (5) Calculate the curl and the divergence of the vector field $\vec{F}(x,y,z)=3z^2\,\vec{\imath}+x\cos(y)\,\vec{\jmath}-2xz\,\vec{k}$.

$$\operatorname{curl}(3z^2\,\vec{\imath}+x\cos(y)\,\vec{\jmath}-2xz\,\vec{k})=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3z^2 & x\cos(y) & -2xz\end{vmatrix}=(0-0)\,\vec{\imath}-(-2z-6z)\,\vec{\jmath}+(\cos(y)-0)\,\vec{k}=$$

$8z\,\vec{\jmath}+\cos(y)\,\vec{k}$ and $\operatorname{div}(3z^2\,\vec{\imath}+x\cos(y)\,\vec{\jmath}-2xz\,\vec{k})=0+x(-\sin(y))-2x=-2x-x\sin(y)$.

VII. (5) Let S be the portion of the cylinder $x^2 + z^2 = 1$ that lies between the vertical planes $y = 0$ and $y = 2 - x$. The surface S is parameterized by $x = \cos(\theta)$, $y = h$, $z = \sin(\theta)$ for $0 \le \theta \le 2\pi$ and $0 \le h \le 2 - \cos(\theta)$.

1. Calculate $\vec{r}_\theta$ and $\vec{r}_h$.

   $\vec{r}_\theta = -\sin(\theta)\,\vec{\imath} + \cos(\theta)\,\vec{k}$ and $\vec{r}_h = \vec{\jmath}$.

2. Calculate $\vec{r}_h \times \vec{r}_\theta$ and $\|\,\vec{r}_h \times \vec{r}_\theta\,\|$.

$$\vec{r}_h \times \vec{r}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{vmatrix} = \cos(\theta)\,\vec{\imath} + \sin(\theta)\,\vec{k}, \text{ so } \|\,\vec{r}_h \times \vec{r}_\theta\,\| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1.$$

VIII. (6) Use Green's Theorem to calculate $\int_C (y^3\,\vec{\imath} - x^3\,\vec{\jmath}) \cdot d\vec{r}$, where C is the circle $x^2 + y^2 = 4$ with the clockwise orientation.

Letting D be the unit disk, and noting that C has the reverse of the positive orientation, Green's Theorem gives us $\int_C (y^3\,\vec{\imath} - x^3\,\vec{\jmath}) \cdot d\vec{r} = -\iint_D \frac{\partial(-x^3)}{\partial x} - \frac{\partial(y^3)}{\partial y}\,dD = \iint_D 3x^2 + 3y^2\,dD = \int_0^{2\pi} d\theta \int_0^2 3r^3\,dr = 2\pi \cdot 12 = 24\pi$.

IX. (6) Calculate $\iint_S (xy\,\vec{\imath} + 4x^2\,\vec{\jmath} + yz\,\vec{k}) \cdot d\vec{S}$, where S is the surface $z = xe^y$, $0 \le x \le 1$, $0 \le y \le 2$.

Using the formula $\iint_S (P\,\vec{\imath} + Q\,\vec{\jmath} + R\,\vec{k}) \cdot d\vec{S} = \iint_D -P\,g_x - Q\,g_y + R\,dD$, we have
$\int_S (xy\,\vec{\imath} + 4x^2\,\vec{\jmath} + yz\,\vec{k}) \cdot d\vec{S} = \int_D -xy\,e^y - 4x^2\,xe^y + yz\,dD = \int_D -xy\,e^y - 4x^2\,xe^y + yxe^y\,dD$
$= \int_D -4x^3 e^y\,dD = -\int_0^2 e^y\,dy \int_0^1 4x^3\,dx = 1 - e^2$.

X. (6) Let S be the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$. Calculate dS in terms of dD, where D is the domain in the xy-plane lying beneath S, and use it to calculate $\iint_S z^2\,dS$.

We calculate $dS = \sqrt{1 + z_x^2 + z_y^2}\;dD = \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dD$
$= \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\,dD = \sqrt{2}\,dD$, so $\iint_S z^2\,dS = \iint_R (\sqrt{x^2+y^2})^2\,\sqrt{2}\,dD$
$= \iint_R r^2\,\sqrt{2}\,r\,dr\,d\theta\,dD = \int_0^{2\pi} d\theta \int_1^2 \sqrt{2}\,r^3\,dr = 2\pi\,\sqrt{2}\,(16-1)/4 = 15\pi/\sqrt{2}$.