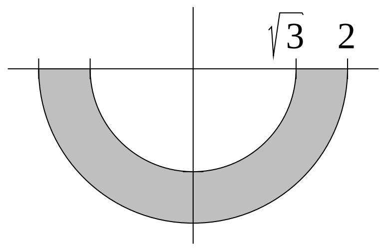
Mathematics 2443-003 Name (please print) ____________________

Examination II Form B

October 18, 2007

Instructions: Give brief answers, but clearly indicate your reasoning.

I. (5) Evaluate by changing to polar coordinates: $\iint_R (x+y)\,dA$, where R is the region that lies below the x-axis and between the circles $x^2+y^2=3$ and $x^2+y^2=4$.



$$\iint_R (x+y)\,dA = \int_\pi^{2\pi}\int_{\sqrt{3}}^{2} (r\cos(\theta)+r\sin(\theta))\,r\,dr\,d\theta = \int_\pi^{2\pi} \frac{r^3}{3}(\cos(\theta)+\sin(\theta))\Big|_{\sqrt{3}}^{2}\,d\theta$$

$$= \int_\pi^{2\pi} \frac{8-3\sqrt{3}}{3}(\cos(\theta)+\sin(\theta))\,d\theta = \frac{8-3\sqrt{3}}{3}(\sin(\theta)-\cos(\theta))\Big|_\pi^{2\pi} = \frac{6\sqrt{3}-16}{3}$$

II. (9) Let E be the upper hemisphere of the unit ball, that is, $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1,\ z \ge 0\}$. For the integral $\iiint_E f(x,y,z)\ dV$, supply the explicit limits of integration, the expression for dV, and (if necessary) the expressions for x, y, and z, that would be needed to calculate the integral:

(i) In xyz-coordinates (x,y,z)

$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$

(ii) In cylindrical coordinates (r,θ,z)

$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} f(r\cos(\theta), r\sin(\theta), z)\,dz\,dr\,d\theta$

(iii) In spherical coordinates (ρ,θ,ϕ)

$\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\theta))\,d\rho\,d\theta\,d\phi$

III. (5) Evaluate the integral $\iint_R e^{y^2}\,dA$, where $R = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$.

$$\iint_R e^{y^2}\,dA = \int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 xe^{y^2}\Big|_0^y\,dy = \int_0^1 ye^{y^2}\,dy = e^{y^2}/2\Big|_0^1 = \frac{e-1}{2}\ .$$

IV. (5) Let E be the solid in the first octant bounded by $x^2+y^2+z^2=1$ and the three coordinate planes (that is, E is the portion of the unit ball that lies in the first octant). Suppose that the density at each point of E equals the distance from the point to the xz-plane. Write integrals to find the mass of E and its moment with respect to the xz-plane. Do not supply explicit limits for the integrals, or try to evaluate the integrals.

The density is $\rho(x,y,z) = y$. The mass and moment are $m = \iiint_E dm = \iiint_E y\ dV$, $M_{xz} = \iiint_E y\,dm = \iiint_E yz\ dV$.

V. (5) Calculate the numerical value of a Riemann sum to estimate the value of $\iint_R x^2y\,dA$, where R is the rectangle $[0,4]\times[0,2]$, i. e. the (x,y) with $0\le x\le 4$ and $0\le y\le 2$. Partition the x-interval $[0,4]$ into two equal subintervals, and partition the y-interval into two equal subintervals, so that the Riemann sum has four terms. Use the Midpoint Rule to choose the sample points.

The rectangles are $[0,2]\times[0,1]$, $[2,4]\times[0,1]$, $[0,2]\times[1,2]$, and $[2,4]\times[1,2]$, and the corresponding midpoints are $(1,1/2)$, $(3,1/2)$, $(1,3/2)$, and $(3,3/2)$. The function values at the midpoints are $1/2$, $9/2$, $3/2$, and $27/2$. Since the area of each rectangle is 2, the Riemann sum is $(1/2)\cdot 2+9/2)\cdot 2+(3/2)\cdot 2+(27/2)\cdot 2=40$.

VI. (5) Sketch a portion of a typical graph $z=f(x,y)$, showing the tangent plane at a point $(x_0,y_0,f(x_0,y_0))$. Let $\vec{v}_x$ be the vector in the tangent plane whose $\vec{\imath}$-component is 1 and whose $\vec{\jmath}$-component is 0 (i. e. $\vec{v}_x$ is a vector of the form $\vec{\imath}+\lambda\vec{k}$ for some number λ). Show $\vec{v}_x$ in your sketch, and express λ in terms of f or its partial derivatives.

For the sketch, see your class notes. λ is $f_x(x_0,y_0)$, so $\vec{v}_x=\vec{\imath}+f_x(x_0,y_0)\vec{k}$.

VII. (5) Calculate $\|\,(\vec{\imath}+f_x(x_0,y_0)\vec{k})\times(\vec{\jmath}+f_y(x_0,y_0)\vec{k})\,\|$. Give the details of the calculation, not just the answer.

$$\|\,(\vec{\imath}+f_x(x_0,y_0)\vec{k})\times(\vec{\jmath}+f_y(x_0,y_0)\vec{k})\,\| = \begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 1 & 0 & f_x(x_0,y_0)\\ 0 & 1 & f_y(x_0,y_0)\end{vmatrix} = \|-f_x(x_0,y_0)\vec{\imath}-f_y(x_0,y_0)\vec{\jmath}+\vec{k}\|$$

$$=\sqrt{(-f_x(x_0,y_0))^2+(-f_y(x_0,y_0))^2+1}=\sqrt{1+f_x(x_0,y_0)^2+f_y(x_0,y_0)^2}$$

VIII. (6) Find the surface area of the portion of the paraboloid $z=x^2+y^2$ that lies above the unit disk in the xy-plane.

We calculate $dS=\sqrt{1+\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2}\,dx\,dy=\sqrt{1+4x^2+4y^2}\,dx\,dy$. Integrating in polar coordinates, the surface area is

$$\int_0^{2\pi}\int_0^1\sqrt{1+4r^2}\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\,\left(\frac{2}{3}\,\frac{(1+4r^2)^{3/2}}{8}\Big|_0^1\right)=\pi\,\frac{5\sqrt{5}-1}{6}\ .$$

IX. (5) Let E be the solid tetrahedron bounded by the coordinate planes and the plane $x+y+2z=2$. Supply limits for the integral $\iiint_E f(x,y,z)\,dV$, assuming that the order of integration is first with respect to x, then with respect to y, then with respect to z.

The top plane is $x=2-y-2z$, and the side in the yz-plane (i. e. where $y=0$) is the triangle bounded by the coordinate axes and the line $y+2z=2$. So the integral is $\int_0^1\int_0^{2-2z}\int_0^{2-y-2z}f(x,y,z)\,dx\,dy\,dz$.

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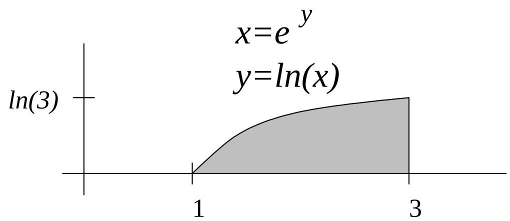
X. (5) Sketch the region and change the order of integration for $\int_1^3 \int_0^{\ln(x)} f(x,y)\,dy\,dx$.



$$\int_0^{\ln(3)} \int_{e^y}^3 f(x,y)\,dx\,dy.$$

XI. (5) Evaluate the integral $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)^{3/2}\,dx\,dy$.

The domain of integration is the right half of the disk of radius a. Changing to polar coordinates, the integral becomes

$$\int_{-\pi/2}^{\pi/2} \int_0^a r^3 \cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{a^5}{5}\,d\theta = \frac{\pi a^5}{5}\ .$$