I. Evaluate by changing to polar coordinates: \( \iint_R (x + y) \, dA \), where \( R \) is the region that lies below the \( x \)-axis and between the circles \( x^2 + y^2 = 3 \) and \( x^2 + y^2 = 4 \).

\[
\int_{\sqrt{3}}^{2} \int_{\theta} \left( r \cos(\theta) + r \sin(\theta) \right) r \, dr \, d\theta = \int_{\pi}^{2\pi} \frac{8 - 3\sqrt{3}}{3} \cos(\theta) + \sin(\theta) \Bigg|_\theta \, d\theta = \frac{6\sqrt{3} - 16}{3}
\]

II. Let \( E \) be the upper hemisphere of the unit ball, that is, \( E = \{ (x, y, z) \mid x^2 + y^2 + z^2 \leq 1, \ z \geq 0 \} \). For the integral \( \iiint_E f(x, y, z) \, dV \), supply the explicit limits of integration, the expression for \( dV \), and (if necessary) the expressions for \( x, y, \) and \( z \), that would be needed to calculate the integral:

(i) In \( xyz \)-coordinates \((x, y, z)\)

\[
\int_{-1}^{1} \int_{\sqrt{1-x^2}} f(x, y, z) \, dz \, dx
\]

(ii) In cylindrical coordinates \((r, \theta, z)\)

\[
\int_{0}^{2\pi} \int_{0}^{1} f(r \cos(\theta), r \sin(\theta), z) \, dz \, dr \, d\theta
\]

(iii) In spherical coordinates \((\rho, \theta, \phi)\)

\[
\int_{0}^{\pi/2} \int_{0}^{2\pi} f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\theta)) \, d\rho \, d\theta \, d\phi
\]

III. Evaluate the integral \( \iint_R e^{y^2} \, dA \), where \( R = \{ (x, y) \mid 0 \leq y \leq 1, \ 0 \leq x \leq y \} \).

\[
\iint_R e^{y^2} \, dA = \int_{0}^{1} \int_{0}^{y} e^{y^2} \, dx \, dy = \int_{0}^{1} x e^{y^2} \bigg|_{0}^{y} \, dy = \int_{0}^{1} ye^{y^2} \, dy = e^{y^2}/2 \bigg|_{0}^{1} = e - \frac{1}{2}.
\]

IV. Let \( E \) be the solid in the first octant bounded by \( x^2 + y^2 + z^2 = 1 \) and the three coordinate planes (that is, \( E \) is the portion of the unit ball that lies in the first octant). Suppose that the density at each point of \( E \) equals the distance from the point to the \( xz \)-plane. Write integrals to find the mass of \( E \) and its moment with respect to the \( xz \)-plane. Do not supply explicit limits for the integrals, or try to evaluate the integrals.

The density is \( \rho(x, y, z) = y \). The mass and moment are \( m = \iiint_E \rho(x, y, z) \, dV \), \( M_{xz} = \iiint_E y \, dV \), \( M_{xz} = \iiint_E y \, dm = \iiint_E yz \, dV \).
V. Calculate the numerical value of a Riemann sum to estimate the value of \( \iint_R x^2 y \, dA \), where \( R \) is the rectangle \([0, 4] \times [0, 2] \), i.e. the \((x, y)\) with \(0 \leq x \leq 4\) and \(0 \leq y \leq 2\). Partition the \(x\)-interval \([0, 4]\) into two equal subintervals, and partition the \(y\)-interval into two equal subintervals, so that the Riemann sum has four terms. Use the Midpoint Rule to choose the sample points.

The rectangles are \([0, 2] \times [0, 1] \), \([2, 4] \times [0, 1] \), \([0, 2] \times [1, 2] \), and \([2, 4] \times [1, 2] \), and the corresponding midpoints are \((1, 1/2) \), \((3, 1/2) \), \((1, 3/2) \), and \((3, 3/2) \). The function values at the midpoints are \(1/2 \), \(9/2 \), \(3/2 \), and \(27/2 \). Since the area of each rectangle is \(2\), the Riemann sum is \((1/2) \cdot 2 + (9/2) \cdot 2 + (3/2) \cdot 2 + (27/2) \cdot 2 = 40\).

VI. Sketch a portion of a typical graph \( z = f(x, y) \), showing the tangent plane at a point \((x_0, y_0, f(x_0, y_0))\).

Let \( \vec{v}_x \) be the vector in the tangent plane whose \(\vec{v}\)-component is \(1\) and whose \(\vec{j}\)-component is \(0\) (i.e. \( \vec{v}_x \) is a vector of the form \( \vec{v} + \lambda \vec{k} \) for some number \(\lambda\)). Show \( \vec{v}_x \) in your sketch, and express \( \lambda \) in terms of \( f \) or its partial derivatives.

For the sketch, see your class notes. \( \lambda \) is \( f_x(x_0, y_0) \), so \( \vec{v}_x = \vec{i} + f_x(x_0, y_0)\vec{k} \).

VII. Calculate \( \| (\vec{i} + f_x(x_0, y_0)\vec{k}) \times (\vec{j} + f_y(x_0, y_0)\vec{k}) \| \). Give the details of the calculation, not just the answer.

\[
\| (\vec{i} + f_x(x_0, y_0)\vec{k}) \times (\vec{j} + f_y(x_0, y_0)\vec{k}) \| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix} = \| f_x(x_0, y_0)\vec{i} - f_y(x_0, y_0)\vec{j} + \vec{k} \|
\]

\[
= \sqrt{(-f_x(x_0, y_0))^2 + (-f_y(x_0, y_0))^2 + 1} = \sqrt{1 + f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2}
\]

VIII. Find the surface area of the portion of the paraboloid \( z = x^2 + y^2 \) that lies above the unit disk in the \(xy\)-plane.

We calculate \( dS = \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dx \, dy = \sqrt{1 + 4x^2 + 4y^2} \, dx \, dy \). Integrating in polar coordinates, the surface area is

\[
\int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \, r \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^1 \left( \frac{2}{3} \frac{(1 + 4r^2)^{3/2}}{8} \right) \left( \frac{1}{0} \right) = \pi \frac{5\sqrt{5} - 1}{6}.
\]

IX. Let \( E \) be the solid tetrahedron bounded by the coordinate planes and the plane \( x + y + 2z = 2 \). Supply limits for the integral \( \iiint_E f(x, y, z) \, dV \), assuming that the order of integration is first with respect to \( x \), then with respect to \( y \), then with respect to \( z \).

The top plane is \( x = 2 - y - 2z \), and the side in the \(yz\)-plane (i.e. where \( y = 0 \)) is the triangle bounded by the coordinate axes and the line \( y + 2z = 2 \). So the integral is \( \int_0^1 \int_0^{2-2z} \int_0^{2-y-2z} f(x, y, z) \, dx \, dy \, dz \).
X. Sketch the region and change the order of integration for \( \int_{1}^{3} \int_{0}^{\ln(x)} f(x, y) \, dy \, dx \).

XI. Evaluate the integral \( \int_{-a}^{a} \int_{0}^{\sqrt{a^2 - y^2}} (x^2 + y^2)^{3/2} \, dx \, dy \).

The domain of integration is the right half of the disk of radius \( a \). Changing to polar coordinates, the integral becomes

\[
\int_{-\pi/2}^{\pi/2} \int_{0}^{a} r^3 \cdot r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{a^5}{5} \, d\theta = \frac{\pi a^5}{5}.
\]