Mathematics 2443-003 Name (please print) ____________________

Examination I Form B

September 18, 2007

Instructions: Give brief answers, but clearly indicate your reasoning. All functions are assumed to have continuous derivatives of all orders, so results such as Clairaut's Theorem may be freely applied if needed.

I. (7) For the function $f(x, y) = y^x$, tell the maximum rate of change of this function at the point $(2, 2)$. Find the rate of change of f at the point $(2, 2)$ in the direction toward $(-1, 3)$.

$\nabla f = y^x \ln(y)\vec{\imath} + xy^{x-1}\vec{\jmath}$, so $\nabla f(2, 2) = 4\ln(2)\vec{\imath} + 4\vec{\jmath}$. The maximum rate of change at $(2, 2)$ is $\| \nabla f(2, 2) \| = 4\sqrt{1 + (\ln(2))^2}$.

The vector from $(2, 2)$ to $(-1, 3)$ is $-3\vec{\imath}+\vec{\jmath}$, so a unit vector in this direction is $\vec{u} = -\dfrac{3}{\sqrt{10}}\vec{\imath}+\dfrac{1}{\sqrt{10}}\vec{\jmath}$. The rate of change of f at $(2, 2)$ in the direction of $\vec{u}$ is $\nabla f(2, 2) \cdot \vec{u} = \dfrac{-3 \cdot 4\ln(2)}{\sqrt{10}} + \dfrac{4}{\sqrt{10}} = \dfrac{4 - 12\ln(2)}{\sqrt{10}}$.

II. (5) Find the domain of the function $f(x, y, z) = e^{\sqrt{z-x^2-y^2}}$. Find its range (that is, the possible values that $f(x, y, z)$ assumes, as one considers all the points (x, y, z) in the domain of f). For finding the range, it may be useful to examine the values of f on the portion of the domain that lies on the z-axis.

We need $z - x^2 - y^2 \geq 0$, or $z \geq x^2 + y^2$. So the domain is the paraboloid $z = x^2 + y^2$ and all the points lying above it (or in set notation, $\{(x, y, z) \mid z \geq x^2 + y^2\}$).

For the range, every value of f is of the form e^t for some $t \geq 0$, so the range lies in the interval $r \geq 1$. On the other hand, if we examine the values of f on the points on the z-axis with $z \geq 0$, we find $f(0, 0, z) = e^{\sqrt{z}}$. Since z can be any non-negative number, these equal the e^r with $r \geq 0$. So all values in the interval $r \geq 1$ do appear as values, i. e. the range is all numbers $r \geq 1$.

III. (6) Sketch a portion of a typical graph $z = f(x, y)$, showing the tangent plane at a point $(x_0, y_0, f(x_0, y_0))$. Let $\vec{v}_x$ be the vector in the tangent plane whose $\vec{\imath}$-component is 1 and whose $\vec{\jmath}$-component is 0 (i. e. $\vec{v}_x$ is a vector of the form $\vec{\imath} + \lambda\vec{k}$ for some number λ). Show $\vec{v}_x$ in your sketch, and express λ in terms of f or its partial derivatives.

For the sketch, see your class notes. λ is $f_x(x_0, y_0)$, so $\vec{v}_x = \vec{\imath} + f_x(x_0, y_0)\vec{k}$.

IV. (5) Calculate the differential $d(x^2 + y^2 + z^2)$. Use it to estimate $(1.1)^2 + 1 + (1.1)^2$ by calculating the linear part of the change of $x^2 + y^2 + z^2$ starting from the point $(1, 1, 1)$.

The differential is $d(x^2 + y^2 + z^2) = 2x\,dx + 2y\,dy + 2z\,dz$.

Evaluating at $x = y = z = 1$, $dx = 0.1$, $dy = 0$, and $dz = 0.1$, we find the linear part of the change to be $2 \cdot (0.1) + 2 \cdot (0) + 2 \cdot (0.1) = 0.4$, so the estimate is 3.4.

V. (5) Tell what Clairaut's Theorem says. Use Clairaut's Theorem to tell why there is no function $f(x, y)$ for which $\dfrac{\partial f}{\partial x} = \sin(xy)$ and $\dfrac{\partial f}{\partial y} = \cos(xy)$.

Clairaut's Theorem says that (under some minor hypotheses on f) $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$, that is, taking the same partial derivatives in different orders yields the same results.

Say you had a function f for which $\dfrac{\partial f}{\partial x} = \sin(xy)$ and $\dfrac{\partial f}{\partial y} = \cos(xy)$. Then, you would have

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}(\sin(xy)) = x\cos(xy)$$

and

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}(\cos(xy)) = -y\,\sin(xy)\ ,$$

which would not be equal, producing a violation of Clairaut's Theorem.

VI. (5) Use the Chain Rule to find $\dfrac{\partial R}{\partial x}$ when $x = 1$ and $y = 2$ if $R(u, v, w) = \ln(u^2 + v^2 + w^2)$, $u = x + 2y$, $v = 2x - y$, and $w = 2xy$.

Using the Chain Rule, we have

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x} = \frac{2u}{u^2+v^2+w^2}\cdot 1 + \frac{2v}{u^2+v^2+w^2}\cdot 2 + \frac{2w}{u^2+v^2+w^2}\cdot 2y\ .$$

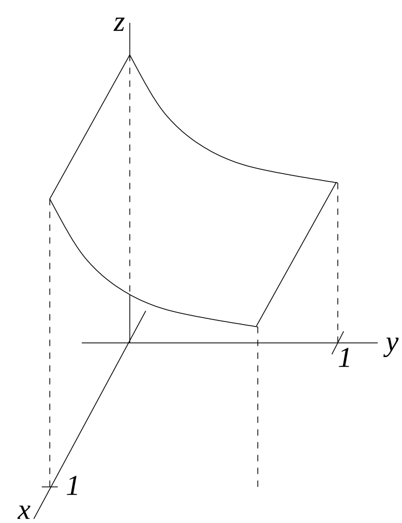
When $x = 1$ and $y = 2$, we have $u = 5$, $v = 0$, and $w = 4$, so

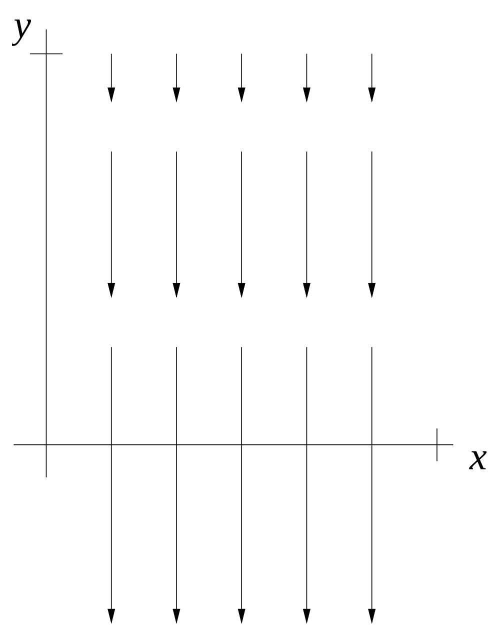
$$\frac{\partial R}{\partial x} = \frac{10}{41}\cdot 1 + \frac{0}{41}\cdot 2 + \frac{8}{41}\cdot 2\cdot 2 = \frac{42}{41}\ .$$

VII. (3) A function R of the variables R_1, R_2, and R_3 is given implicitly by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. Use *implicit differentiation* to find $\dfrac{\partial R}{\partial R_1}$, $\dfrac{\partial R}{\partial R_2}$, and $\dfrac{\partial R}{\partial R_3}$.

Applying $\dfrac{\partial}{\partial R_i}$, we have $-\dfrac{1}{R^2}\dfrac{\partial R}{\partial R_i} = -\dfrac{1}{R_i^2}$, so $\dfrac{\partial R}{\partial R_i} = \dfrac{R^2}{R_i^2}$.

VIII. (4) In an xy-coordinate system, sketch the gradient of the function whose graph is shown to the right.





IX. (4) Find all critical points of the function $f(x,y) = x^4 + y^4 - 4xy + 2$.

We have $\dfrac{\partial f}{\partial x} = 4x^3 - 4y$ and $\dfrac{\partial f}{\partial y} = 4y^3 - 4x$. To find the critical points, we solve $4x^3 - 4y = 0$ and $4y^3 - 4x = 0$. The first says that $y = x^3$, and putting this in the second we find that $x^9 - x = 0$. One solution is $x = 0$, and if $x \neq 0$ then $x^8 = 1$ so x is 1 or -1. Since $y = x^3$, the critical points are $(-1,-1)$, $(0,0)$, and $(1,1)$.

X. (5) Let T be the triangle bounded by the x-axis, the y-axis, and the line $x + y = 1$. Find the maximum and minimum values of $f(x,y) = 2x^2 + y^2$ on:

(a) The bottom side of T, i. e. the side that lies in the x-axis.

The points on the bottom side of the triangle are the $(x,0)$ with $0 \le x \le 1$. On these points, $f(x,0) = 2x^2$, so the minimum is 0 at $(0,0)$ and the maximum is 2 at $(1,0)$.

(b) The diagonal side of T, i. e. the side that lies in the line $x + y = 1$.

The points on the diagonal side of the triangle are the $(x, 1-x)$ with $0 \le x \le 1$. On these points, $f(x, 1-x) = 2x^2 + (1-x)^2 = 3x^2 - 2x + 1$. The critical point is where $6x - 2 = 0$, i. e. $x = 1/3$, so the extrema can only occur at $(0,1)$, $(1/3, 2/3)$, and $(1,0)$. The values of f at these points are 1, 2/3, and 2, so the minimum of f on this edge is 2/3 at $(1/3, 2/3)$ and the maximum is 2 at $(1,0)$.

XI. (5) Use Lagrange multipliers to find the extreme value or values of $f(x,y) = 2x^2 + y^2$ on the line $x + y = 1$.

The constraint function is $g(x,y) = x + y$. We have $\nabla f = 4x\vec{\imath} + 2y\vec{\jmath}$ and $\nabla g = \vec{\imath} + \vec{\jmath}$. The equation $\nabla f = \lambda \nabla g$ becomes $4x^2\vec{\imath} + y^2\vec{\jmath} = \lambda\vec{\imath} + \lambda\vec{\jmath}$, or $4x = \lambda$, $2y = \lambda$. This says $y = 2x$, and since $x + y = 1$ we have $x + 2x = 1$ or $x = 1/3$ and $y = 2/3$. So the only local extreme value is the minimum at $(1/3, 2/3)$ that we found in the previous problem.