Mathematics 2443-003 Name (please print) ____________________

Final Examination Form A

December 12, 2007

Instructions: Give brief, clear answers.

I. (4) Evaluate by changing to polar coordinates: $\iint_R x + y\, dR$ where R is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ and above the x-axis.

$$\iint_R x + y\, dR = \int_0^{\pi}\int_1^{\sqrt{2}} (r\cos(\theta) + r\sin(\theta))\, r\, dr\, d\theta = \int_0^{\pi} \cos(\theta) + \sin(\theta)\, d\theta \int_1^{\sqrt{2}} r^2\, dr$$
$$= 2(2\sqrt{2} - 1)/3.$$

II. (6) For the function $f(x, y) = \ln(x^2+y^2)$, find the maximum rate of change at the point $(1, 2)$, and the direction in which it occurs. Find the directional derivative of f at $(1, 2)$ in the direction toward $(2, 4)$.

We have $\nabla f(x, y) = \frac{2x}{x^2+y^2}\vec{\imath} + \frac{2y}{x^2+y^2}\vec{\jmath}$, so $\nabla f(1, 2) = \frac{2}{5}\vec{\imath} + \frac{4}{5}\vec{\jmath}$.

The maximum rate of change at $(1, 2)$ is $\|\nabla f(1, 2)\| = \|\frac{2}{5}\vec{\imath} + \frac{4}{5}\vec{\jmath}\| = \frac{2}{\sqrt{5}}$, and it is in the direction of $\nabla f(1, 2)$.

A vector in the direction from $(1, 2)$ toward $(2, 4)$ is $\vec{\imath} + 2\vec{\jmath}$, so a unit vector in that direction is $\vec{u} = \frac{1}{\sqrt{5}}\vec{\imath} + \frac{2}{\sqrt{5}}\vec{\jmath}$. The rate of change in that direction is $\nabla f(1, 2) \cdot \vec{u} = \frac{2}{\sqrt{5}}$.

(Alternatively, one can observe that the given direction is exactly the direction of $\nabla f(1, 2)$, so the rate of change in that direction, as already found, is $\|\nabla f(1, 2)\| = \frac{2}{\sqrt{5}}$.)

III. (6) Let S be the portion of the sphere of radius a that lies in the first octant. Use the standard parameterization of S to calculate $\iint_S (y\,\vec{\imath} - x\,\vec{\jmath} + \vec{k}\,) \cdot d\vec{S}$.

If R is the region $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$ in the $\theta\phi$-plane, then

$$\iint_S (y\,\vec{\imath} - x\,\vec{\jmath} + \vec{k}) \cdot d\vec{S} = \iint_R (y\,\vec{\imath} - x\,\vec{\jmath} + \vec{k}) \cdot (\vec{r}_\phi \times \vec{r}_\theta)\, dR = \iint_R (y\,\vec{\imath} - x\,\vec{\jmath} + \vec{k}) \cdot a\,\sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})\, dR$$
$$= \iint_R a\sin(\phi) z\, dR = \int_0^{\pi/2} d\theta \int_0^{\pi/2} a\sin(\phi) a\cos(\phi)\, d\phi = (\pi/2)a^2\sin^2(\phi)/2\Big|_0^{\pi/2} = \pi a^2/4$$

IV. (5) Use the Divergence Theorem to calculate the surface integral $\iint_S (x^2z^3\,\vec{\imath} + 2xyz^3\,\vec{\jmath} + xz^4\,\vec{k}\,) \cdot d\vec{S}$, where S is the surface of the box with $0 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 1$.

$$\iint_S (x^2z^3\,\vec{\imath} + 2xyz^3\,\vec{\jmath} + xz^4\,\vec{k}\,) \cdot d\vec{S} = \iiint_E 2xz^3 + 2xz^3 + 4xz^3\, dV$$
$$= \int_0^1\int_0^2\int_0^3 8xz^3\, dx\, dy\, dz = \int_0^3 2x\, dx \int_0^2 dy \int_0^1 4z^3\, dz = 9 \cdot 2 \cdot 1 = 18\ .$$

V. (4) The radius of a right circular cone is increasing at a rate of 6 in/s while its height is decreasing at a rate of 3 in/s. At what rate is the volume $V = \pi r^2 h/3$ changing when the radius is 10 and the height is 5?

Using the Chain Rule, $\dfrac{dV}{dt} = \dfrac{\partial(\pi r^2h/3)}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial(\pi r^2h/3)}{\partial h}\dfrac{dr}{dt} = \dfrac{2\pi rh}{3}\dfrac{dr}{dt} + \dfrac{\pi r^2}{3}\dfrac{dh}{dt}$. Evaluating when $r = 10$ and $h = 5$, we find $\dfrac{dV}{dt} = \dfrac{100\pi}{3} \cdot 6 + \dfrac{100\pi}{3} \cdot (-3) = 100\pi$. It is increasing at a rate of 100π in^3/s.

VI. (6) Let S be the upper half of the sphere of radius 2, that is, the points (x, y, z) with $x^2 + y^2 + z^2 = 4$ and $z \geq 0$, and suppose that S is oriented with the upward normal. Use Stokes' Theorem to evaluate $\iint_S \mathrm{curl}(x^2 e^{yz}\,\vec{\imath} + y^2 e^{xz}\,\vec{\jmath} + z^2 e^{xy}\,\vec{k}) \cdot d\vec{S}$.

The boundary of S is the circle C of radius 2 in the xy-plane, with the positive orientation. By Stokes' Theorem, $\iint_S \mathrm{curl}(x^2 e^{yz}\,\vec{\imath} + y^2 e^{xz}\,\vec{\jmath} + z^2 e^{xy}\,\vec{k}) \cdot d\vec{S} = \int_C (x^2 e^{yz}\,\vec{\imath} + y^2 e^{xz}\,\vec{\jmath} + z^2 e^{xy}\,\vec{k}) \cdot d\vec{r}$. On C, $z = 0$, so this line integral is $\int_C (x^2\,\vec{\imath} + y^2\,\vec{\jmath}) \cdot d\vec{r}$. Using Green's Theorem, $\int_C (x^2\,\vec{\imath} + y^2\,\vec{\jmath}) \cdot d\vec{r} = \iint_D \frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial y}\, dD = 0$.

VII. (4) Let S be the upper half of the sphere of radius 1, that is, the points (x, y, z) with $x^2 + y^2 + z^2 = 1$ and $z \geq 0$. Using the geometric interpretation of the surface integral of a vector field as the "flux" (that is, not by calculation using a parameterization or a formula from the formulas list), explain each of the following equalities:

1. $\iint_S \vec{\jmath} \cdot d\vec{S} = 0$

   Consider the flow of unit speed in the $\vec{\jmath}$ direction. $\iint_S \vec{\jmath} \cdot d\vec{S} = 0$ is the net flow (per unit time) across S. Since this flow is horizontal, the amount that flows into the hemisphere on the left side exactly equals the amount that flows out of it on the right, so the net flow is 0.

2. $\iint_S \vec{k} \cdot d\vec{S} = \pi$

   Consider the flow of unit speed in the $\vec{k}$ direction. $\iint_S \vec{k} \cdot d\vec{S} = 2\pi$ is the net flow per unit time across S. Since the flow is directly upward, the amount that flows across S is the same as the amount that flows across the the unit disk in the xy-plane. The amount of vertical flow per unit time is just the area of this disk, which is π.

VIII. (4) Verify that the function $u = \cos(x - at) + \ln(x + at)$ is a solution to the wave equation $u_{tt} = a^2 u_{xx}$.

$(\cos(x - at) + \ln(x + at))_{tt} = \Big(a \sin(x - at) + \frac{a}{x + at}\Big)_t = -a^2 \cos(x - at) - \frac{a^2}{(x + at)^2}$ and $(\cos(x - at) + \ln(x + at))_{xx} = \Big(-\sin(x - at) + \frac{1}{x + at}\Big)_x = -\cos(x - at) - \frac{1}{(x + at)^2}$, so $a^2 u_{xx} = a^2\Big(-\cos(x - at) - \frac{1}{(x + at)^2}\Big) = -a^2 \cos(x - at) - \frac{a^2}{(x + at)^2} = u_{tt}$.

IX. (10) Let S be the portion of the cylinder $x^2+z^2=4$ that lies between the vertical planes $y=0$ and $y=2-x$. The surface S is parameterized by $x=2\cos(\theta)$, $y=h$, $z=2\sin(\theta)$ for $0\le\theta\le 2\pi$ and $0\le h\le 2-2\cos(\theta)$.

1. Calculate $\vec r_\theta$, $\vec r_h$, $\vec r_h\times\vec r_\theta$, and $\|\,\vec r_h\times\vec r_\theta\,\|$.

$\vec r_\theta=-2\sin(\theta)\,\vec\imath+2\cos(\theta)\,\vec k$ and $\vec r_h=\vec\jmath$.

$$\vec r_h\times\vec r_\theta=\begin{vmatrix}\vec\imath & \vec\jmath & \vec k\\ 0 & 1 & 0\\ -2\sin(\theta) & 0 & 2\cos(\theta)\end{vmatrix}=2\cos(\theta)\,\vec\imath+2\sin(\theta)\,\vec k,\text{ so }\|\,\vec r_h\times\vec r_\theta\,\|=\sqrt{4\cos^2(\theta)+4\sin^2(\theta)}=2.$$

2. Calculate $\displaystyle\iint_S x\,dS$.

$dS=\|\,\vec r_h\times\vec r_\theta\,\|\,dR=2\,dR$, where R is the region $0\le\theta\le2\pi$ and $0\le h\le 2-\cos(\theta)$ in the parameter domain. So we have

$$\iint_S x\,dS=\iint_R\cos(\theta)\,2\,dR=\int_0^{2\pi}\int_0^{2-2\cos(\theta)}2\cos(\theta)\,dh\,d\theta=\int_0^{2\pi}2\cos(\theta)(2-2\cos(\theta))\,d\theta$$
$$=\int_0^{2\pi}4\cos(\theta)-2(1+\cos(2\theta))\,d\theta=0-4\pi+0=-4\pi\ .$$

3. Calculate $\displaystyle\iint_S x\,\vec k\cdot d\vec S$.

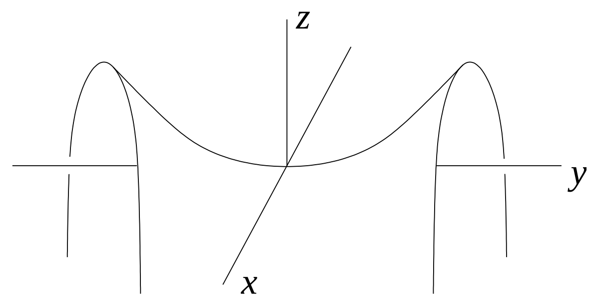
$$\iint_S x\,\vec k\cdot d\vec S=\iint_R x\,\vec k\cdot(\vec r_h\times\vec r_\theta)\,dR=\iint_R x\,\vec k\cdot(2\cos(\theta)\,\vec\imath+2\sin(\theta)\,\vec k)\,dR=\iint_R 2\sin(\theta)\cos(\theta)\,dR$$
$$=\int_0^{2\pi}\int_0^{2-\cos(\theta)}2\sin(\theta)\cos(\theta)\,dh\,d\theta=\int_0^{2\pi}2\sin(\theta)\cos(\theta)(2-\cos(\theta))\,d\theta$$
$$=\int_0^{2\pi}4\sin(\theta)\cos(\theta)-2\sin(\theta)\cos^2(\theta)\,d\theta=2\sin^2(\theta)-2\cos^3(\theta)/3\,\Big|_0^{2\pi}=0\ .$$

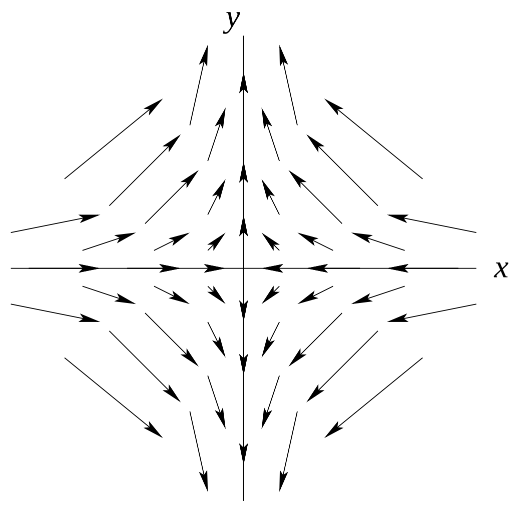
X. (6) The curl of the vector field $y\,\vec\imath-z\,\vec\jmath+x\,\vec k$ is $\vec\imath-\vec\jmath-\vec k$. Let S be the triangle which is the part of the plane $2x+y+z=2$ that lies in the first octant. Give S the upward normal, and give its boundary C the corresponding positive orientation. Use Stokes' Theorem to evaluate the line integral $\displaystyle\int_C(y\,\vec\imath-z\,\vec\jmath+x\,\vec k)\cdot d\vec r$. (Hint: the surface integral on S is easy to calculate if one uses the definition $\displaystyle\iint_S\vec G\cdot d\vec S=\iint_S\vec G\cdot\vec n\,dS$.)

A normal vector to S is $2\vec\imath+\vec\jmath+\vec k$. This is upward, since the $\vec k$-component is positive, and $\|\,2\vec\imath+\vec\jmath+\vec k\,\|=\sqrt6$, so a unit upward normal is $\vec n=(1/\sqrt6)\,(2\vec\imath+\vec\jmath+\vec k)$. Using Stokes' Theorem, we have

$$\int_C(y\,\vec\imath+z\,\vec\jmath+x\,\vec k)\cdot d\vec r=\iint_S\operatorname{curl}(y\,\vec\imath+z\,\vec\jmath+x\,\vec k)\cdot d\vec S=\iint_S(\vec\imath-\vec\jmath-\vec k)\cdot\vec n\,dS$$
$$=\iint_S(\vec\imath-\vec\jmath-\vec k)\cdot((1/\sqrt6)(2\vec\imath+\vec\jmath+\vec k))\,dS=\iint_S(1/\sqrt6)(2-1-1)\,dS=0\ .$$

XI. (4) In an xy-coordinate system, sketch the gradient of the function whose graph is shown to the right.





XII. (3) A function R of the variables R_1, R_2, and R_3 is given implicitly by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$. Use *implicit differentiation* to find $\dfrac{\partial R}{\partial R_3}$.

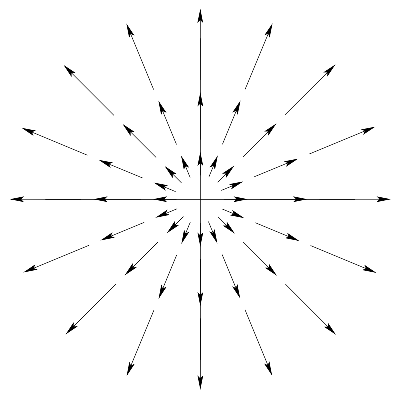
Applying $\dfrac{\partial}{\partial R_3}$, we have $-\dfrac{1}{R^2}\,\dfrac{\partial R}{\partial R_3} = -\dfrac{1}{R_3^2}$, so $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$.

XIII. (5) Use the Divergence Theorem to show that if E is a solid with boundary the surface S, then $\displaystyle\iint_S \left(\frac{x}{3}\vec{\imath} + \frac{y}{3}\vec{\jmath} + \frac{z}{3}\vec{k}\right)\cdot d\vec{S}$ always equals the volume of E.

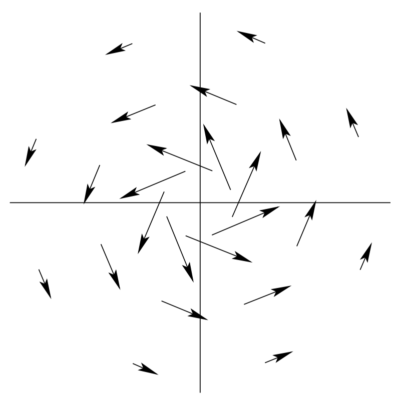
$$\iint_S \left(\frac{x}{3}\vec{\imath} + \frac{y}{3}\vec{\jmath} + \frac{z}{3}\vec{k}\right)\cdot d\vec{S} = \iiint_E \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\,dV = \iiint_E dV = \mathrm{vol}(E).$$

XIV. On two different coordinate systems, graph the following vector fields:
(6)

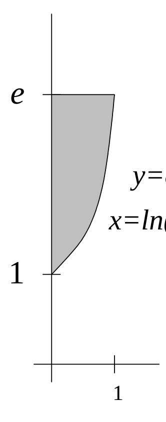
1. $\vec{F}(x,y) = x\,\vec{\imath} + y\,\vec{\jmath}$



2. $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\,\vec{\imath} + \dfrac{x}{x^2+y^2}\,\vec{\jmath}$



XV. Sketch the region and change the order of integration for $\displaystyle\int_0^1 \int_{e^x}^{e} f(x,y)\,dy\,dx$.
(4)



$$\int_1^e \int_0^{\ln(y)} f(x,y)\,dx\,dy.$$