I. Find an equation for the sphere that has center \((4, -2, \pi)\) and contains the origin.

The radius is the distance from \((4, -2, \pi)\) to the origin \((0, 0, 0)\), which is \(\sqrt{4^2 + (-2)^2 + \pi^2} = \sqrt{\pi^2 + 20}\), so the equation is \((x - 4)^2 + (y + 2)^2 + (z - \pi)^2 = \pi^2 + 20\).

II. Find parametric equations for the line that is the intersection of the planes \(x - 2y - 3z = 1\) and \(2x + y + z = 1\).

The normal vectors are \(\vec{i} - 2\vec{j} - 3\vec{k}\) and \(2\vec{i} + \vec{j} + \vec{k}\). The line lies in both planes, so its direction vectors are perpendicular to both normal vectors. Therefore one possible direction vector is \((\vec{i} - 2\vec{j} - 3\vec{k}) \times (2\vec{i} + \vec{j} + \vec{k})\). We calculate this to be \(-3\vec{i} - 7\vec{j} + 5\vec{k}\). To find a point on the intersection line, we just need one solution to the simultaneous equations \(x - 2y - 3z = 1\) and \(2x + y + z = 1\). When, say, \(x = 0\), the solution would have to satisfy \(-2y - 3z = 1\) and \(y + z = 1\), giving \(y = 4\) and \(z = -3\), so the point \((0, 4, -3)\) lies in the intersection. Therefore parametric equations are \(x = t, y = 4 - 7t, z = -3 + 5t\).

III. Determine the convergence or divergence of each of these series, using any information or method other than the Limit Comparison Test.

1. \(\sum_{n=1}^{\infty} \frac{7 + 7^n}{8 + 8^n}\)

   We have \(\frac{7 + 7^n}{8 + 8^n} < \frac{7^n}{8^n} < \left(\frac{7}{8}\right)^n\). The series \(\sum \frac{2 \cdot 7^n}{8^n}\) is geometric with \(r = \frac{7}{8} < 1\), so \(\sum_{n=1}^{\infty} \frac{7 + 7^n}{8 + 8^n}\) converges by the Comparison Test.

2. \(\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n\)

   Since \(\lim_{n \to \infty} \left(\frac{n-1}{n}\right)^n = e^{-1} < 1\), \(\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n\) converges by the Root Test.

IV. Carry out a translation of the form \(X = x - x_0, Y = y - y_0, Z = z - z_0\) to put the equation \(x^2 - y^2 - 4z^2 = 2x + 16z + 16\) into standard form. (You do not have to draw the graph, but you can if you want.) The graph is a hyperboloid of two sheets. Which traces are ellipses? At what points (in \(X\)\(Y\)\(Z\)-coordinates) does it meet the \(X\)-axis?

Completing the square gives the equation \((x - 1)^2 - y^2 - 4(z + 2)^2 = 1\). If we take \((X, Y, Z)\)-coordinates centered at \((1, 0, -2)\), that is, \(X = x - 1, Y = y,\) and \(Z = z + 2\), the equation becomes \(X^2 - Y^2 - 4Z^2 = 1\). This is a hyperboloid of two sheets meeting the \(X\)-axis in two points. Writing the equation as \(Y^2 + 4Z^2 = X^2 - 1\), we see that the traces with \(X = k\) are ellipses when \(|X| > 1\). When \(X = \pm 1\), we obtain \(Y^2 + 4Z^2 = 0\), which is the origin, so these are the points where the surface meets the \(X\)-axis. (Alternatively, you can just say that it meets the \(X\)-axis when \(Y = Z = 0\), giving \(X^2 = 1\) or \(X = \pm 1\).)
V. Let $\vec{a} = -4\vec{i} + \vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + \vec{j} - \vec{k}$.

1. Calculate the scalar projection of $\vec{a}$ onto $\vec{b}$.

$$\frac{\vec{a} \cdot \vec{b}}{\| \vec{b} \|} = \frac{(-4\vec{i} + \vec{j} + 3\vec{k}) \cdot (\vec{i} + \vec{j} - \vec{k})}{\sqrt{1 + 1 + 1}} = \frac{(-4) \cdot 1 + 1 \cdot 1 + 3 \cdot (-1)}{\sqrt{3}} = \frac{-6}{\sqrt{3}} = -2\sqrt{3}$$

2. Calculate the vector projection of $\vec{a}$ onto $\vec{b}$.

$$\frac{\vec{a} \cdot \vec{b}}{\| \vec{b} \|} \vec{b} = \frac{(-4\vec{i} + \vec{j} + 3\vec{k}) \cdot (\vec{i} + \vec{j} - \vec{k})}{(\vec{i} + \vec{j} - \vec{k}) \cdot (\vec{i} + \vec{j} - \vec{k})} (\vec{i} + \vec{j} - \vec{k}) = \frac{-6}{3} (\vec{i} + \vec{j} - \vec{k}) = -2\vec{i} - 2\vec{j} + 2\vec{k}$$

VI. Give an algebraic verification that $\| \vec{a} + \vec{b} \|^2 + \| \vec{a} - \vec{b} \|^2 = 2 \| \vec{a} \|^2 + 2 \| \vec{b} \|^2$, but not by doing a lengthy calculation involving $\vec{i}$, $\vec{j}$, and $\vec{k}$.

$$\| \vec{a} + \vec{b} \|^2 + \| \vec{a} - \vec{b} \|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) + (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = 2 \| \vec{a} \|^2 + 2 \| \vec{b} \|^2$$

VII. Give examples of the following:

1. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$.

$$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}, \text{ but } (\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}.$$ 

2. Nonzero vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ but $\vec{b} \neq \vec{c}$.

$$\vec{i} \times (\vec{i} + \vec{j}) = \vec{i} \times \vec{i} + \vec{i} \times \vec{j} = \vec{i} \times \vec{j}, \text{ but } \vec{i} + \vec{j} \neq \vec{j}.$$ 

VIII. Here is a fact: Let $\vec{u}$ be any unit vector (i.e. $\| \vec{u} \| = 1$). If $\vec{v}$ is any vector, then the length of $\vec{v} \times \vec{u}$ is no more than the length of $\vec{v}$.

1. Give an algebraic explanation for this fact.

$$\| \vec{v} \times \vec{u} \| = \| \vec{v} \| \| \vec{u} \| \sin(\theta) = \| \vec{v} \| \sin(\theta) \leq \| \vec{v} \|.$$ 

2. Give a geometric explanation for this fact.

$$\| \vec{v} \times \vec{u} \|$$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{u}$. This area is the base times the height. Regarding $\vec{v}$ as the “base,” the height is no more than the length of the other side $\vec{u}$, so the area $\| \vec{v} \times \vec{u} \|$ is no more than $\| \vec{v} \|$ times 1.
IX. If you saw a movie of the planes \( y \cos(\theta) + z \sin(\theta) = 0 \) as \( \theta \) went from 0 to \( 2\pi \), what would they look like?

(4) Use both words and picture(s) in your explanation.

Normal vectors to these planes are \( \cos(\theta) \hat{j} + \sin(\theta) \hat{k} \), which are the position vectors of the unit circle in the \( yz \)-plane. All the planes contain the origin, so also contain the \( x \)-axis. For \( \theta = 0 \), the normal vector is \( \hat{j} \), so the plane would be the \( xz \)-plane. As \( \theta \) increases, we would see the plane rotating counterclockwise, as indicated in the figure below. When \( \theta = \pi \), we would see the \( yz \)-plane again, although with the original sides interchanged. When \( \theta \) reaches \( 2\pi \), the plane would be rotated back to its original position.

![Diagram](image)

X. Derive these formulas expressing rectangular coordinates in terms of spherical coordinates:

\[ \begin{align*}
    x &= \rho \sin(\phi) \cos(\theta), \\
    y &= \rho \sin(\phi) \sin(\theta), \\
    z &= \rho \cos(\phi).
\end{align*} \]

(4) From the right triangle shown in the figure, we read off \( z = \rho \cos(\phi) \) and \( r = \rho \sin(\phi) \). Then, using the formula for polar coordinates in the horizontal plane containing \( P \), we have \( x = \rho \sin(\phi) \cos(\theta) \) and \( x = \rho \sin(\phi) \sin(\theta) \).

XI. Draw the graph of this equation given in spherical coordinates: \( \rho = \phi^3 \).

(4)
XII. Write a possible equation for this saddle surface:

\[ y = z^2 \] and \[ y = -x^2 \], so a possible equation would be \[ y = z^2 - x^2 \].

XIII. In higher dimensions, say dimension \( n \), there are vectors \( \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \) that play the roles of \( \vec{i}, \vec{j}, \) and \( \vec{k} \). In particular, \( \vec{e}_i \cdot \vec{e}_j = 0 \) when \( i \neq j \), and \( \vec{e}_i \cdot \vec{e}_i = 1 \) for each \( i \). Verify that if an \( n \)-dimensional vector \( \vec{v} \) equals \( r_1 \vec{e}_1 + r_2 \vec{e}_2 + \cdots + r_n \vec{e}_n \), then \( r_1 = \vec{v} \cdot \vec{e}_1 \).

We calculate \( \vec{v} \cdot \vec{e}_1 = (r_1 \vec{e}_1 + r_2 \vec{e}_2 + \cdots + r_n \vec{e}_n) \cdot \vec{e}_1 = r_1 \vec{e}_1 \cdot \vec{e}_1 + r_2 \vec{e}_2 \cdot \vec{e}_1 + \cdots + r_n \vec{e}_n \cdot \vec{e}_1 = r_1 \cdot 1 + r_2 \cdot 0 + \cdots + r_n \cdot 0 = r_1. \)