Mathematics 2433-001H Name (please print) ____________________

Examination III

November 15, 2007

I. (4) Find an equation for the sphere that has center $(4, -2, \pi)$ and contains the origin.

The radius is the distance from $(4, -2, \pi)$ to the origin $(0, 0, 0)$, which is $\sqrt{4^2 + (-2)^2 + \pi^2} = \sqrt{\pi^2 + 20}$, so the equation is $(x - 4)^2 + (y + 2)^2 + (z - \pi)^2 = \pi^2 + 20$.

II. (4) Find parametric equations for the line that is the intersection of the planes $x-2y-3z = 1$ and $2x+y+z = 1$.

The normal vectors are $\vec{\imath}-2\vec{\jmath}-3\vec{k}$ and $2\vec{\imath}+\vec{\jmath}+\vec{k}$. The line lies in both planes, so its direction vectors are perpendicular to both normal vectors. Therefore one possible direction vector is $(\vec{\imath}-2\vec{\jmath}-3\vec{k})\times(2\vec{\imath}+\vec{\jmath}+\vec{k})$. We calculate this to be $-3\vec{\imath}-7\vec{\jmath}+5\vec{k}$. To find a point on the intersection line, we just need one solution to the simultaneous equations $x - 2y - 3z = 1$ and $2x + y + z = 1$. When, say, $x = 0$, the solution would have to satisfy $-2y - 3z = 1$ and $y + z = 1$, giving $y = 4$ and $z = -3$, so the point $(0, 4, -3)$ lies in the intersection. Therefore parametric equations are $x = t$, $y = 4 - 7t$, and $z = -3 + 5t$.

III. (6) Determine the convergence or divergence of each of these series, using any information or method *other than the Limit Comparison Test.*

1. $\displaystyle\sum_{n=1}^{\infty} \frac{7 + 7^n}{8 + 8^n}$

We have $\dfrac{7 + 7^n}{8 + 8^n} < \dfrac{7 + 7^n}{8^n} < \dfrac{7^n + 7^n}{8^n} = \dfrac{2 \cdot 7^n}{8^n}$. The series $\sum \dfrac{2 \cdot 7^n}{8^n}$ is geometric with $r = \frac{7}{8} < 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{7 + 7^n}{8 + 8^n}$ converges by the Comparison Test.

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$

Since $\displaystyle\lim_{n\to\infty} \left(\left(\frac{n-1}{n}\right)^{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n-1}{n}\right)^{n} = \lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^{n} = e^{-1} < 1$, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$ converges by the Root Test.

IV. (6) Carry out a translation of the form $X = x - x_0$, $Y = y - y_0$, $Z = z - z_0$ to put the equation $x^2 - y^2 - 4z^2 = 2x + 16z + 16$ into standard form. (You do not have to draw the graph, but you can if you want.) The graph is a hyperboloid of two sheets. Which traces are ellipses? At what points (in XYZ-coordinates) does it meet the X-axis?

Completing the square gives the equation $(x - 1)^2 - y^2 - 4(z + 2)^2 = 1$. If we take (X, Y, Z)-coordinates centered at $(1, 0, -2)$, that is, $X = x - 1$, $Y = y$, and $Z = z + 2$, the equation becomes $X^2 - Y^2 - 4Z^2 = 1$. This is a hyperboloid of two sheets meeting the X-axis in two points. Writing the equation as $Y^2 + 4Z^2 = X^2 - 1$, we see that the traces with $X = k$ are ellipses when $|X| > 1$. When $X = \pm 1$, we obtain $Y^2 + 4Z^2 = 0$, which is the origin, so these are the points where the surface meets the X-axis. (Alternatively, you can just say that it meets the X-axis when $Y = Z = 0$, giving $X^2 = 1$ or $X = \pm 1$.)

V. (4) Let $\vec{a} = -4\vec{\imath} + \vec{\jmath} + 3\vec{k}$ and $\vec{b} = \vec{\imath} + \vec{\jmath} - \vec{k}$.

1. Calculate the scalar projection of $\vec{a}$ onto $\vec{b}$.

$$\vec{a} \cdot \frac{\vec{b}}{\|\vec{b}\|} = (-4\vec{\imath} + \vec{\jmath} + 3\vec{k}) \cdot \frac{\vec{\imath} + \vec{\jmath} - \vec{k}}{\sqrt{1+1+1}} = \frac{(-4)\cdot 1 + 1 \cdot 1 + 3 \cdot (-1)}{\sqrt{3}} = \frac{-6}{\sqrt{3}} = -2\sqrt{3}$$

2. Calculate the vector projection of $\vec{a}$ onto $\vec{b}$.

$$\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b} = \frac{(-4\vec{\imath} + \vec{\jmath} + 3\vec{k}) \cdot (\vec{\imath} + \vec{\jmath} - \vec{k})}{(\vec{\imath} + \vec{\jmath} - \vec{k}) \cdot (\vec{\imath} + \vec{\jmath} - \vec{k})}\,(\vec{\imath} + \vec{\jmath} - \vec{k}) = \frac{-6}{3}\,(\vec{\imath} + \vec{\jmath} - \vec{k}) = -2\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$$

VI. (4) Give an algebraic verification that $\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2 = 2\,\|\vec{a}\|^2 + 2\,\|\vec{b}\|^2$, but not by doing a lengthy calculation involving $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$.

$$\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) + (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b} = 2\,\vec{a}\cdot\vec{a} + 2\,\vec{b}\cdot\vec{b} = 2\,\|\vec{a}\|^2 + 2\,\|\vec{b}\|^2$$

VII. (4) Give examples of the following:

1. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$.

$\vec{\imath}\times(\vec{\imath}\times\vec{\jmath}) = \vec{\imath}\times\vec{k} = -\vec{\jmath}$, but $(\vec{\imath}\times\vec{\imath})\times\vec{\jmath} = \vec{0}\times\vec{\jmath} = \vec{0}$.

2. Nonzero vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ but $\vec{b}\neq\vec{c}$.

$\vec{\imath}\times(\vec{\imath}+\vec{\jmath}) = \vec{\imath}\times\vec{\imath}+\vec{\imath}\times\vec{\jmath} = \vec{\imath}\times\vec{\jmath}$, but $\vec{\imath}+\vec{\jmath}\neq\vec{\jmath}$.

VIII. (4) Here is a fact: Let $\vec{u}$ be any unit vector (i. e. $\|\vec{u}\| = 1$). If $\vec{v}$ is any vector, then the length of $\vec{v}\times\vec{u}$ is no more than the length of $\vec{v}$.

1. Give an algebraic explanation for this fact.

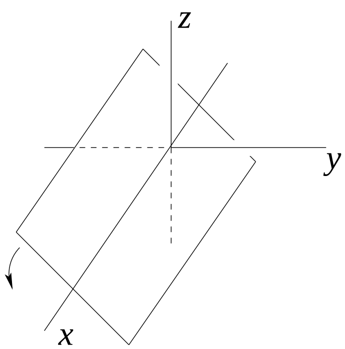
$\|\vec{v}\times\vec{u}\| = \|\vec{v}\|\,\|\vec{u}\|\,\sin(\theta) = \|\vec{v}\|\,\sin(\theta) \le \|\vec{v}\|$.

2. Give a geometric explanation for this fact.

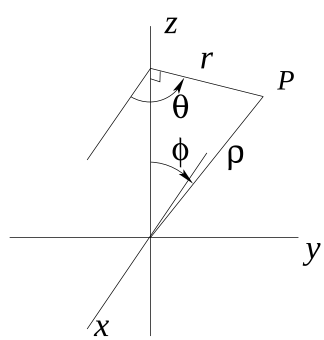
$\|\vec{v}\times\vec{u}\|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{u}$. This area is the base times the height. Regarding $\vec{v}$ as the "base," the height is no more than the length of the other side $\vec{u}$, so the area $\|\vec{v}\times\vec{u}\|$ is no more than $\|\vec{v}\|$ times 1.

IX. If you saw a movie of the planes $y\cos(\theta) + z\sin(\theta) = 0$ as θ went from 0 to 2π, what would they look like?
(4) Use both words and picture(s) in your explanation.

Normal vectors to these planes are $\cos(\theta)\vec{\jmath} + \sin(\theta)\vec{k}$, which are the position vectors of the unit circle in the yz-plane. All the planes contain the origin, so also contain the x-axis. For $\theta = 0$, the normal vector is $\vec{\jmath}$, so the plane would be the xz-plane. As θ increases, we would see the plane rotatiing counterclockwise, as indicated in the figure below. When $\theta = \pi$, we would see the yz-plane again, although with the original sides interchanged. When θ reaches 2π, the plane would be rotated back to its original position.

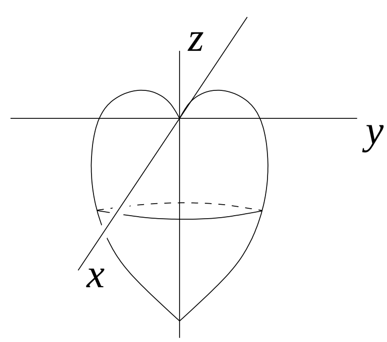


X. Derive these formulas expressing rectangular coordinates in terms of spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$,
(4) $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.

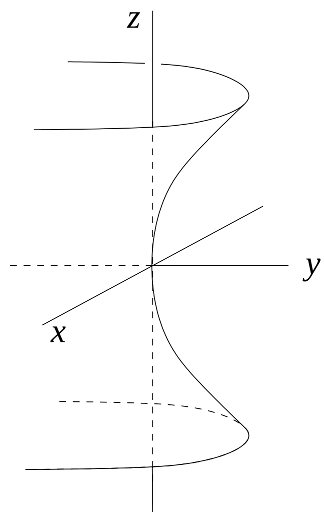


From the right triangle shown in the figure, we read off $z = \rho\cos(\phi)$ and $r = \rho\sin(\phi)$. Then, using the formula for polar coordinates in the horizontal plane containing P, we have $x = \rho\sin(\phi)\cos(\theta)$ and $x = \rho\sin(\phi)\sin(\theta)$.

XI. Draw the graph of this equation given in spherical coordinates: $\rho = \phi^3$.
(4)



XII. (4) Write a possible equation for this saddle surface:



It is a saddle surface linear in y. In the yz-plane, it looks like $y = z^2$, and in the xy-plane, like $y = -x^2$, so a possible equation would be $y = z^2 - x^2$.

XIII. (4) In higher dimensions, say dimension n, there are vectors $\vec{e}_1$, $\vec{e}_2, \ldots,$ $\vec{e}_n$ that play the roles of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$. In particular, $\vec{e}_i \cdot \vec{e}_j = 0$ when $i \neq j$, and $\vec{e}_i \cdot \vec{e}_i = 1$ for each i. Verify that if an n-dimesional vector $\vec{v}$ equals $r_1\vec{e_1} + r_2\vec{e_2} + \cdots + r_n\vec{e_n}$, then $r_1 = \vec{v} \cdot \vec{e}_1$.

We calculate $\vec{v} \cdot \vec{e}_1 = (r_1\vec{e_1} + r_2\vec{e_2} + \cdots + r_n\vec{e_n}) \cdot \vec{e}_1 = r_1 \ \vec{e_1} \cdot \vec{e}_1 + r_2 \ \vec{e_2} \cdot \vec{e}_1 + \cdots + \cdot r_n \ \vec{e_n} \cdot \vec{e}_1 = r_1 \cdot 1 + r_2 \cdot 0 + \cdots + \cdot r_n \cdot 0 = r_1$.