Mathematics 2433-001H Name (please print) ______________________

Examination II

October 23, 2007

I. (12) Determine the convergence or divergence of each of these series, using any information or method *other than the Limit Comparison Test.* If the series has some negative terms, check for absolute convergence as well.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n\ \ln(n)}$

   The sequence $\left\{\dfrac{1}{n\ln(n)}\right\}$ is decreasing for $n \geq 2$, since both n and $\ln(n)$ increase as n increases. Also, $\lim \dfrac{1}{n\ \ln(n)} = 0$, say by squeezing between the sequences $\{0\}$ and $\{\frac{1}{n}\}$. By the Alternating Series Test, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n\ \ln(n)}$ converges.

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ \ln(n)}$

   As in the previous problem, the function $\dfrac{1}{x\ \ln(x)}$ is decreasing for $x \geq 2$. We have $\displaystyle\int_2^{\infty} \frac{1}{x\ \ln(x)}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\ \ln(x)}\,dx = \lim_{b\to\infty} \ln(\ln(x))\Big|_2^b = \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(2)) = \infty$. So $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ \ln(n)}$ diverges by the Integral Test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{7+7^n}{8+8^n}$

   We have $\dfrac{7+7^n}{8+8^n} < \dfrac{7+7^n}{8^n} < \dfrac{7^n+7^n}{8^n} = \dfrac{2\cdot 7^n}{8^n}$. The series $\sum \dfrac{2\cdot 7^n}{8^n}$ is geometric with $r = \frac{7}{8} < 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{7+7^n}{8+8^n}$ converges by the Comparison Test.

4. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$

   Since $\displaystyle\lim_{n\to\infty} \left(\left(\frac{n-1}{n}\right)^{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^n = e^{-1} < 1$, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$ converges by the Root Test.

II. (6) Find the Maclaurin series for $\arctan(x)$ by using the fact that $\arctan(x) = \int \frac{1}{1+x^2}\,dx$.

When $-1 < x^2 < 1$, i. e. $-1 < x < 1$, we have

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx \int \frac{1}{1-(-x^2)}\,dx = \int \sum_{n=0}^{\infty} (-x^2)^n\,dx$$
$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

When $x = 0$, we have $\arctan(0) = C + 0$, so $C = 0$. Therefore $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $-1 < x < 1$.

[This is a beautiful series: $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. In particular, when $x = 1$, this becomes $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.]

III. (8) Analyze the convergence behavior of the power series $\sum_{n=0}^{\infty} \frac{n}{4^n}(3-x)^n$. In particular, determine its center, radius of convergence, and for every real number x determine whether the series converges absolutely, converges conditionally, or diverges.

First we rewrite the series as $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$, and note that the center is $a = 3$. Now, to apply the Ratio Test, we calculate

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}(n+1)|x-3|^{n+1}}{4^{n+1}}\right|}{\left|\frac{(-1)^n n|x-3|^n}{4^n}\right|} = \lim_{n\to\infty} \frac{n}{4(n+1)}\ |x-3| = \frac{1}{4}\ |x-3|\ .$$

This is less than 1 exactly when $-1 \le \frac{1}{4}(x-3) < 1$, that is, $-1 < x < 7$. So the radius of convergence is 4, and the series converges absolutely when $-1 < x < 7$ and diverges when $x < -1$ or $7 < x$. It remains to check the endpoints. When $x = -1$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n = \sum_{n=0}^{\infty} n$, which diverges since the terms do not limit to 0. When $x = 4$, it becomes $\sum_{n=0}^{\infty} (-1)^n n$, which diverges for the same reason.

IV. (6) Using the Maclaurin series of $\cos(x)$, find the Maclaurin series of the following functions. Make reasonable simplifications.

(i) $\cos(2x)$

Replacing x with $2x$ in $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, we obtain $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n}$.

(ii) $\sin^2(x)$

Using the identity $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$, we have

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{2n} = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n-1} 4^n}{2\,(2n)!} x^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4^n}{2\,(2n)!} x^{2n}\ .$$

V. (6) Let $\sum a_n$ be a series with positive terms. Suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ with $0 < L < 1$.

1. Let r be a number with $L < r < 1$. Explain (at least informally) why for all sufficiently large values of n, say $n \geq N$, each $\sqrt[n]{a_n} < r$.

   Taking $\epsilon = r - L$, the definition of limit says that there is some N so that whenever $n \geq N$, $|\sqrt[n]{a_n} - L| < r - L$, which says that $L - r < \sqrt[n]{a_n} - L < r - L$ and hence $\sqrt[n]{a_n} < r$. (Informally, for all sufficiently large n, the distance from $\sqrt[n]{a_n}$ to L must be smaller than $r - L$, forcing $\sqrt[n]{a_n} < L + r - L = r$.)

2. Use the Comparison test to deduce that $\sum a_n$ converges.

   From $\sqrt[n]{a_n} < r$, we have $0 < a_n < r^n$ for all sufficiently large n, so $\sum a_n$ converges by comparison with the geometric series $\sum r^n$.

VI. (5) Showing a reasonable amount of detail, use integration by parts to verify that $\int_a^b \frac{(b-t)^5}{5!} f^{(5)}(t)\,dt = \frac{f^{(6)}(a)}{6!}(b-a)^6 + \int_a^b \frac{(b-t)^6}{6!} f^{(6)}(t)\,dt$.

We use integration by parts with $u = f^{(5)}(t)$ and $dv = \frac{(b-t)^5}{5!}$. Using $du = f^{(6)}(t)\,dt$ and $v = -\frac{(b-t)^6}{6\cdot 5!} = -\frac{(b-t)^6}{6\cdot 6!}$, we find

$$\int_a^b \frac{(b-t)^5}{5!} f^{(6)}(t)\,dt = -\frac{(b-t)^6}{6!} f^{(6)}(t)\Big|_a^b - \int_a^b -\frac{(b-t)^6}{6!} f^{(6)}(t)\,dt$$

$$= 0 - \left(-\frac{(b-a)^6}{6!} f^{(6)}(a)\right) + \int_a^b -\frac{(b-t)^6}{6!} f^{(6)}(t)\,dt = \frac{f^{(6)}(a)}{6!}(b-a)^6 + \int_a^b \frac{(b-t)^6}{6!} f^{(6)}(t)\,dt$$

VII. (5) Recall that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (this can be seen, for example, by using the Ratio Test to check that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for every value of x and deducing that its terms limit to 0). Use Taylor's Theorem $R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$ to verify that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x > 0$. (Hint: $e^t \leq e^x$ for all t with $0 \leq t \leq x$.)

Fix an $x > 0$. Applying Taylor's Theorem with $f(x) = e^x$ and $a = 0$, we estimate

$$|R_n(x)| = \left|\int_0^x \frac{(x-t)^n}{n!} e^t\,dt\right| \leq \left|\int_0^x \frac{(x-t)^n}{n!} e^x\,dt\right| = e^x \left|\int_0^x \frac{(x-t)^n}{n!}\,dt\right| = e^x \left| -\frac{(x-t)^{n+1}}{(n+1)!}\Big|_0^x \right| = e^x \frac{x^{n+1}}{(n+1)!}$$

Since e^x is fixed and $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, this shows that $\lim_{n\to\infty} |R_n(x)| = 0$ and hence $\lim_{n\to\infty} R_n(x) = 0$. Since $R_n(x) = f(x) - P_n(x)$ and $P_n(x)$ is a partial sum of the Taylor series, we conclude that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

[When $x < 0$, the only difference is that $e^t < 1$ for $x \leq t \leq 0$, so $|R_n(x)| \leq \frac{x^{n+1}}{(n+1)!}$. Again, $\lim R_n(x) = 0$ and consequently $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.]

VIII. (5) Recall that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. Use Lagrange's form $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ to verify that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x > 0$.

Since all derivatives of e^x are equal to e^x, Lagrange's form for $R_n(x)$ tells us that $R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$ for some c between 0 and x. Since e^x is increasing, we have $e^c < e^x$. Consequently we have for all n that

$$|R_n(x)| \leq \frac{e^x x^n}{n!} \ .$$

For a fixed x, e^x is just some number, and $\lim \frac{x^n}{n!} = 0$, so $\lim |R_n(x)| = 0$ and hence $\lim R_n(x) = 0$. Therefore we have for each $x > 0$ that

$$e^x = \lim P_n(x) + R_n(x) = \lim P_n(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ .$$

[When $x < 0$, the only difference is that $e^c < 1$, since c lies between x and 0. Again, $\lim R_n(x) = 0$ and consequently $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.]

IX. (5) Evaluate $\int_0^x e^{-t^2}\, dt$.

$$\int_0^x e^{-t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-t^2)}{n!}\, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)n!}\Bigg|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}.$$