Mathematics 2433-001H Name (please print) ____________________

Examination II

October 23, 2007

I. (12) Determine the convergence or divergence of each of these series, using any information or method *other than the Limit Comparison Test.* If the series has some negative terms, check for absolute convergence as well.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n\ln(n)}$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{7+7^n}{8+8^n}$

4. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$

II. (6) Find the Maclaurin series for $\arctan(x)$ by using the fact that $\arctan(x) = \displaystyle\int \frac{1}{1+x^2}\,dx$.

III. (8) Analyze the convergence behavior of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{4^n}(3-x)^n$. In particular, determine its center, radius of convergence, and for every real number x determine whether the series converges absolutely, converges conditionally, or diverges.

IV. (6) Using the Maclaurin series of $\cos(x)$, find the Maclaurin series of the following functions. Make reasonable simplifications.

(i) $\cos(2x)$

(ii) $\sin^2(x)$

V. (6) Let $\sum a_n$ be a series with positive terms. Suppose that $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} = L$ with $0 < L < 1$.

1. Let r be a number with $L < r < 1$. Explain (at least informally) why for all sufficiently large values of n, say $n \geq N$, each $\sqrt[n]{a_n} < r$.

2. Use the Comparison test to deduce that $\sum a_n$ converges.

VI. (5) Showing a reasonable amount of detail, use integration by parts to verify that $\displaystyle\int_a^b \frac{(b-t)^5}{5!} f^{(5)}(t)\,dt = \frac{f^{(6)}(a)}{6!}(b-a)^6 + \int_a^b \frac{(b-t)^6}{6!} f^{(6)}(t)\,dt.$

VII. (5) Recall that $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (this can be seen, for example, by using the Ratio Test to check that the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for every value of x and deducing that its terms limit to 0). Use Taylor's Theorem $R_n(x) = \displaystyle\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$ to verify that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x > 0$. (Hint: $e^t \leq e^x$ for all t with $0 \leq t \leq x$.)

VIII. Recall that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. Use Lagrange's form $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ to verify that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x > 0$.
(5)

IX. Evaluate $\int_0^x e^{-t^2}\,dt$.
(5)