Mathematics 2433-001H

Examination I

September 20, 2007

Instructions: Give concise answers, but clearly indicate your reasoning.

I. A curve is given parametrically by the equations
   \[ x = \int_0^t \cos(\pi u^2/2) \, du, \quad y = \int_0^t \sin(\pi u^2/2) \, du. \]
   Find the length of the portion of this curve with \( 0 \leq t \leq \pi \).
   
   \[
   \frac{dx}{dt} = \cos(\pi t^2/2) \quad \text{and} \quad \frac{dy}{dt} = \sin(\pi t^2/2),
   \]
   so
   \[
   ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2(\pi t^2/2) + \sin^2(\pi t^2/2)} \, dt = \frac{dt}{\pi}.
   \]
   Thus the desired length is \( \int_0^\pi \frac{dt}{\pi} = \frac{\pi}{2} \).

II. An equation \( r = f(\theta) \) defines a polar curve. Use the Chain Rule
   \[
   \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin(\theta) + r \cos(\theta)}{\frac{dr}{d\theta} \cos(\theta) - r \sin(\theta)}
   \]
   to derive a general formula for \( \frac{dy}{dx} \) in terms of \( r \) and \( \theta \) for such a curve.

III. A curve given by the parametric equations \( x = 2t^3, \quad y = 1 - t^2, \quad -\infty < t < \infty \).
   Find the area of the region bounded by the curve and the \( x \)-axis.
   \[
   \text{We have} \ y \geq 0 \text{ exactly when} \ -1 \leq t \leq 1, \text{so we want the area between the curve and the} \ x \text{-axis \ for these} \ t \text{-values. We calculate it as} \ 
   \int_{t=-1}^{t=1} y \, dx = \int_{t=-1}^{t=1} (1 - t^2) \, d(2t^3) = \int_{t=-1}^{t=1} (1 - t^2) \, d(6t^2) \, dt = \int_{-1}^{1} 6t^2 - 6t^4 \, dt = 4t^3 - \frac{6t^5}{5} \bigg|_{-1}^{1} = 4 - \frac{12}{5} = \frac{8}{5}.
   \]

IV. Find the surface area of a sphere of radius \( R \) by regarding it as \( x = R \cos(\theta), \quad y = R \sin(\theta) \) and rotating
   about the \( x \)-axis.
   
   We have \( ds = R \, d\theta \), and the distance to the axis of rotation is \( \rho = y = R \sin(\theta) \). So the surface area, integrating from \( \theta = \pi \) to \( \theta = 0 \) so as to integrate in the direction of increasing \( x \), is
   \[
   \int_{\pi}^{0} 2\pi R \sin(\theta) \, R \, d\theta = 2\pi R^2 \int_{\pi}^{0} \sin(\theta) \, d\theta = -2\pi R^2 \int_{\pi}^{0} \cos(\theta) \, d\theta = -2\pi R^2 (-1 - 1) = 4\pi R^2.
   \]

V. Calculate the area of the region that lies inside the polar curve \( r = 4 \sin(\theta) \) and outside the polar curve \( r = 2 \).
   
   We have \( 4 \sin(\theta) \geq 2 \) when \( \sin(\theta) \geq \frac{1}{2} \), that is, \( \pi/6 \leq \theta \leq 5\pi/6 \). So the desired area is
   \[
   \int_{\pi/6}^{5\pi/6} \frac{1}{2}(16 \sin^2(\theta) - 4) \, d\theta = \int_{\pi/6}^{5\pi/6} 2 - 4 \cos(2\theta) \, d\theta = 2\theta - 2 \sin(2\theta) \bigg|_{\pi/6}^{5\pi/6} = 10\pi/6 - 2 \sin(5\pi/3) - (2\pi/6 - 2 \sin(\pi/3)) = 5\pi/3 + \sqrt{3} - \pi/3 + \sqrt{3} = 4\pi/3 + 2\sqrt{3}.
   \]
VI. The graph of a certain equation $r = f(\theta)$ is shown at the right, in a rectangular $\theta$-$r$ coordinate system. In an $x$-$y$ coordinate system, make a reasonably accurate graph of the polar equation $r = f(\theta)$ for this function.

VII. State the Squeeze Theorem. Use the Squeeze Theorem to find the limit of $\left\{\frac{(2n-1)!}{(2n+1)!}\right\}$ by comparing it to the sequence $\{0\}$ and to some sequence of the form $\{n^p\}$.

If $a_n \leq b_n \leq c_n$ for all $n$ (or, for all sufficiently large $n$), and $\lim a_n = L = \lim c_n$, then $\lim b_n$ exists and equals $L$.

We have $\frac{(2n-1)!}{(2n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots 2n + 1} = \frac{1}{2n(2n+1)}$. Also, $0 \leq \frac{1}{2n(2n+1)} \leq \frac{1}{n \cdot n} = n^{-2}$. Since $\lim 0 = 0$ and $\lim n^{-2} = 0$, the Squeeze Theorem shows that $\lim \left\{\frac{(2n-1)!}{(2n+1)!}\right\}$ exists and equals 0.

VIII. Determine whether each of the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \arctan(n)$
   
   $\lim \arctan(n) = \pi/2 \neq 0$, so the series diverges.

2. $\sum_{n=1}^{\infty} (\sin(1))^n$
   
   The series is geometric with $r = \sin(1)$ lying in the range $-1 < r < 1$, so the series converges.

IX. Find all $x$ for which the series $\sum_{n=0}^{\infty} \frac{1}{x^n}$ converges.

   The series is geometric with $r = \frac{1}{x}$, so it will converge exactly when $-1 < \frac{1}{x} < 1$, that is, when $x < -1$ or $1 < x$. 
X. State the Monotonicity Theorem. Analyze the convergence of the sequence \( \left\{ \frac{n}{n^2 + 1} \right\} \) as follows:

1. State the Monotonicity Theorem.

A bounded monotonic sequence of real numbers converges to some real number.

2. Calculate that the derivative of the function \( \frac{x}{x^2 + 1} \) is nonpositive when \( x \geq 1 \). Deduce that \( \left\{ \frac{n}{n^2 + 1} \right\} \) is decreasing.

The derivative is \( \frac{(x^2 + 1) \cdot 1 - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} \). For \( x \geq 1 \), \( 1 - x^2 \leq 0 \) so this function is negative and therefore \( \frac{x}{x^2 + 1} \) is decreasing. In particular, its values at the integers, which are the terms of the sequence \( \left\{ \frac{n}{n^2 + 1} \right\} \), are decreasing.

3. Verify any other hypotheses of the Monotonicity Theorem, to deduce that \( \left\{ \frac{n}{n^2 + 1} \right\} \) converges.

For each \( n \), we have \( 0 \leq \frac{n}{n^2 + 1} \leq \frac{n^2 + 1}{n^2 + 1} = 1 \), so the sequence terms are bounded between 0 and 1. Applying the Monotonicity Theorem, we deduce that the sequence is convergent.

4. Now, find the limit by dividing numerator and denominator by \( n \) and observing the effect of letting \( n \to \infty \).

\[
\frac{n}{n^2 + 1} = \frac{1}{n + \frac{1}{n}}.
\]
Since \( \frac{1}{n} \to 0 \), the denominator \( n + \frac{1}{n} \to \infty \), while the numerator is always 1, so \( \frac{1}{n + \frac{1}{n}} \to 0 \)

XI. Use a simple diagram involving \( dr \) and \( d\theta \) to derive an expression for \( ds \) in terms of \( dr \) and \( d\theta \).

\[
\begin{align*}
Y \\
\vdots \\
X
\end{align*}
\]

From this diagram, we have \( ds^2 = (r
d\theta)^2 + dr^2 \), so

\[
ds = \sqrt{r^2 d\theta^2 + dr^2} = \sqrt{r^2 d\theta^2 + \left( \frac{dr}{d\theta} \right)^2} = \sqrt{r^2 d\theta^2 + \left( \frac{dr}{d\theta} \right)^2 d\theta^2} = \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} d\theta.
\]