Mathematics 2433-001H Name (please print) ______________________

Examination I

September 20, 2007

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (4) A curve is give parametrically by the equations $x = \int_0^t \cos(\pi u^2/2)\,du$, $y = \int_0^t \sin(\pi u^2/2)\,du$. Find the length of the portion of this curve with $0 \le t \le \pi$.

$\frac{dx}{dt} = \cos(\pi t^2/2)$ and $\frac{dy}{dt} = \sin(\pi t^2/2)$, so $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{\cos^2(\pi t^2/2) + \sin^2(\pi t^2/2)}\,dt = dt$. So the desired length is $\int_0^\pi dt = \pi$.

II. (4) An equation $r = f(\theta)$ defines a polar curve. Use the Chain Rule $\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}$ to derive a general formula for $\frac{dy}{dx}$ in terms of r and θ for such a curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d(r\sin(\theta)}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

III. (4) A curve given by the parametric equations $x = 2t^3$, $y = 1 - t^2$, $-\infty < t < \infty$. Find the area of the region bounded by the curve and the x-axis.

We have $y \ge 0$ exactly when $-1 \le t \le 1$, so we want the area between the curve and the x-axis for these t-values. We calculate it as $\int_{t=-1}^{t=1} y\,dx = \int_{-1}^{1} (1 - t^2)\,d(2t^3)\,dt = \int_{-1}^{1} (1 - t^2)\,d(6t^2)\,dt = \int_{-1}^{1} 6t^2 - 6t^4\,dt = 4t^3 - \frac{6}{5}t^5\big|_{-1}^{1} = 4 - \frac{12}{5} = \frac{8}{5}$.

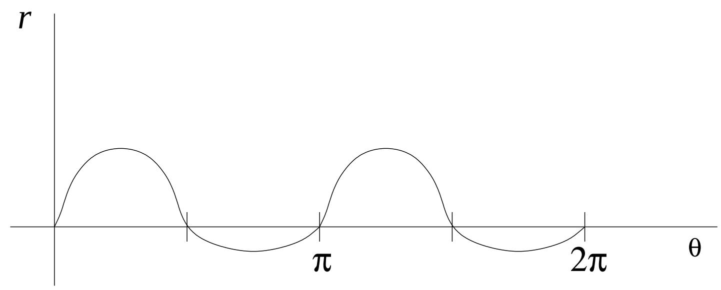
IV. (4) Find the surface area of a sphere of radius R by regarding it as $x = R\cos(\theta)$, $y = R\sin(\theta)$ and rotating about the x-axis.

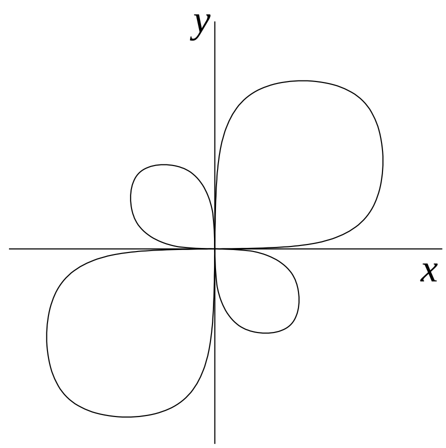
We have $ds = R\,d\theta$, and the distance to the axis of rotation is $\rho = y = R\sin(\theta)$. So the surface area, integrating from $\theta = \pi$ to $\theta = 0$ so as to integrate in the direction of increasing x, is $\int_\pi^0 2\pi R\sin(\theta)\,R\,d\theta = 2\pi R^2 \int_\pi^0 \sin(\theta)\,d\theta = -2\pi R^2 \int_0^\pi \cos(\theta)\big|_\pi^0 = -2\pi R^2(-1 - 1) = 4\pi R^2$.

V. (4) Calculate the area of the region that lies inside the polar curve $r = 4\sin(\theta)$ and outside the polar curve $r = 2$.

We have $4\sin(\theta) \ge 2$ when $\sin(\theta) \ge \frac{1}{2}$, that is, $\pi/6 \le \theta \le 5\pi/6$. So the desired area is $\int_{\pi/6}^{5\pi/6} \frac{1}{2}(16\sin^2(\theta) - 4)\,d\theta = \int_{\pi/6}^{5\pi/6} 2 - 4\cos(2\theta)\,d\theta = 2\theta - 2\sin(2\theta)\big|_{\pi/6}^{5\pi/6} = 10\pi/6 - 2\sin(5\pi/3) - (2\pi/6 - 2\sin(\pi/3)) = 5\pi/3 + \sqrt{3} - \pi/3 + \sqrt{3} = 4\pi/3 + 2\sqrt{3}$.

VI. (4) The graph of a certain equation $r = f(\theta)$ is shown at the right, in a rectangular θ-r coordinate system. In an x-y coordinate system, make a reasonably accurate graph of the polar equation $r = f(\theta)$ for this function.





VII. (5) State the Squeeze Theorem. Use the Squeeze Theorem to find the limit of $\left\{\dfrac{(2n-1)!}{(2n+1)!}\right\}$ by comparing it to the sequence $\{0\}$ and to some sequence of the form $\{n^p\}$.

If $a_n \leq b_n \leq c_n$ for all n (or, for all sufficiently large n), and $\lim a_n = L = \lim c_n$, then $\lim b_n$ exists and equals L.

We have $\dfrac{(2n-1)!}{(2n+1)!} = \dfrac{1 \cdot 2 \cdot 3 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots 2n+1} = \dfrac{1}{2n(2n+1)}$. Also, $0 \leq \dfrac{1}{2n(2n+1)} \leq \dfrac{1}{n \cdot n} = n^{-2}$. Since $\lim 0 = 0$ and $\lim n^{-2} = 0$, the Squeeze Theorem shows that $\lim \left\{\dfrac{(2n-1)!}{(2n+1)!}\right\}$ exists and equals 0.

VIII. (4) Determine whether each of the following series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \arctan(n)$

   $\lim \arctan(n) = \pi/2 \neq 0$, so the series diverges.

2. $\displaystyle\sum_{n=1}^{\infty} (\sin(1))^n$

   The series is geometric with $r = \sin(1)$ lying in the range $-1 < r < 1$, so the series converges.

IX. (4) Find all x for which the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{x^n}$ converges.

The series is geometric with $r = \frac{1}{x}$, so it will converge exactly when $-1 < \frac{1}{x} < 1$, that is, when $x < -1$ or $1 < x$.

X. (8) State the Monotonicity Theorem. Analyze the convergence of the sequence $\left\{ \dfrac{n}{n^2+1} \right\}$ as follows:

1. State the Monotonicity Theorem.

   A bounded monotonic sequence of real numbers converges to some real number.

2. Calculate that the derivative of the function $\dfrac{x}{x^2+1}$ is nonpositive when $x \geq 1$. Deduce that $\left\{ \dfrac{n}{n^2+1} \right\}$ is decreasing.

   The derivative is $\dfrac{(x^2+1)\cdot 1 - x \cdot 2x}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$. For $x \geq 1$, $1 - x^2 \leq 0$ so this function is negative and therefore $\dfrac{x}{x^2+1}$ is decreasing. In particular, its values at the integers, which are the terms of the sequence $\left\{ \dfrac{n}{n^2+1} \right\}$, are decreasing.

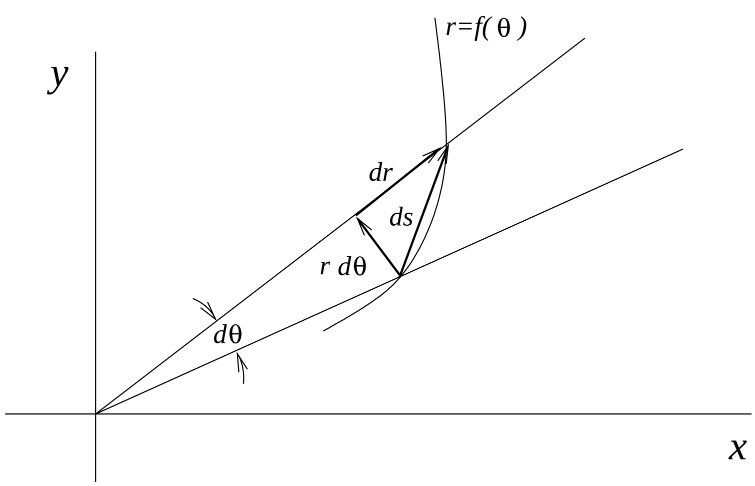
3. Verify any other hypotheses of the Monotonicity Theorem, to deduce that $\left\{ \dfrac{n}{n^2+1} \right\}$ converges.

   For each n, we have $0 \leq \dfrac{n}{n^2+1} \leq \dfrac{n^2+1}{n^2+1} = 1$, so the sequence terms are bounded between 0 and 1. Applying the Monotonicity Theorem, we deduce that the sequence is convergent.

4. Now, find the limit by dividing numerator and denominator by n and observing the effect of letting $n \to \infty$.

   $\dfrac{n}{n^2+1} = \dfrac{1}{n + \frac{1}{n}}$. Since $\frac{1}{n} \to 0$, the denominator $n + \frac{1}{n} \to \infty$, while the numerator is always 1, so $\dfrac{1}{n+\frac{1}{n}} \to 0$

XI. (5) Use a simple diagram involving dr and $d\theta$ to derive an expresssion for ds in terms of dr and $d\theta$.



From this diagram, we have $ds^2 = (r\,d\theta)^2 + dr^2$, so

$$ds = \sqrt{r^2\,d\theta^2 + dr^2} = \sqrt{r^2\,d\theta^2 + \left(\frac{dr}{d\,\theta}\right)^2} = \sqrt{r^2\,d\theta^2 + \left(\frac{dr}{d\,\theta}\right)^2 d\theta^2} = \sqrt{r^2 + \left(\frac{dr}{d\,\theta}\right)^2}\ d\theta.$$