Mathematics 2433-003H   Name (please print) ______________________

Final Examination

December 11, 2007

I. (7) Analyze the convergence behavior of the power series $\sum_{n=1}^{\infty} \frac{1}{nb^n}(x-a)^n$, where a and b are constants with $b > 0$. That is, determine its center, radius of convergence, and for every real number x determine whether the series converges absolutely, converges conditionally, or diverges.

The center is a. Applying the Ratio Test, we have $\lim_{n\to\infty} \dfrac{\frac{1}{(n+1)\,b^{n+1}}\,|x-a|^{n+1}}{\frac{1}{n\,b^n}|x-a|^n} = \lim_{n\to\infty} \dfrac{n|x-a|}{(n+1)b} =$ $|x-a|/b$. We have $-1 < |x-a|/b < 1$ exactly when $-b < x-a < b$, so the radius of convergence is b. Therefore the series converges absolutely for $a-b < x < a+b$ and diverges when $x < a-b$ or $a+b < x$.

It remains to check the endpoints. For $x = a-b$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{nb^n}(-b)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges conditionally.

For $x = a-b$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{nb^n} b^n = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

II. (7) State the Comparison Test, and use it to verify that $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges. (Hint: First verify that $\lim_{n\to\infty} n^{1/n} = 1$.)

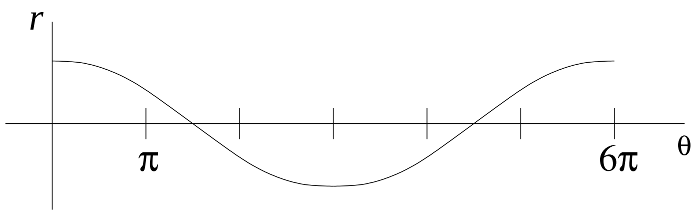
The Comparison Test states that if $\sum a_n$ and $\sum b_n$ are series with positive terms, and $a_n < b_n$ for all n (or at least for all sufficiently large n), then

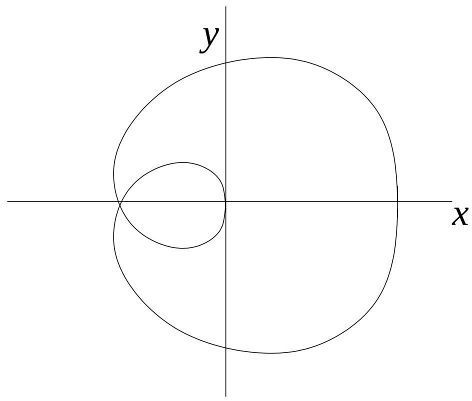
If $\sum b_n$ converges, then $\sum a_n$ converges.

If $\sum a_n$ diverges, then $\sum b_n$ diverges.

We first note that $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln(n)/n} = e^0 = 1$, so for sufficiently large values of n, we have $n^{1/n} < 2$. So for sufficiently large n, we have $\frac{1}{n^{1+1/n}} = \frac{1}{n\,n^{1/n}} > \frac{1}{2n}$. Since $\sum \frac{1}{2n}$ diverges, the Comparison Test shows that $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.

III. (5) Graph the equation $r = \cos(\theta/3)$ for $0 \le \theta \le 6\pi$, first in the θ-r plane, then as a polar equation in the x-y plane.





(The graph is traced out twice as θ goes from 0 to 6π.)

IV. (7) State the Limit Comparison Test, and use it to verify that $\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)$ diverges. (Hints: Use L'Hôpital's Rule to compare it to $\sum \frac{1}{n}$. You may need the facts that $\lim_{n\to\infty} 2^{1/n} = 1$ and $\frac{d(a^x)}{dx} = a^x \ln(a)$.)

The Limit Comparison Test says that if $\sum a_n$ and $\sum b_n$ are series with positive terms, and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ for some number c with $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Using l'Hôpital's Rule, we compute

$$\lim_{n\to\infty} \frac{\sqrt[n]{2}-1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln(2)\,2^{1/n}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \ln(2)\,2^{1/n} = \ln(2)\ .$$

Since $\sum \frac{1}{n}$ diverges, the Limit Comparison Test shows that $\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)$ diverges.

V. Give examples of the following:
(7)

1. A divergent series whose terms limit to 0.

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

2. A conditionally convergent series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

3. A geometric series $\sum_{n=0}^{\infty} r^n$ that converges to π.

We know that $\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$ for $-1 < r < 1$, so we just need the value of r so that $\dfrac{1}{1-r} = \pi$. This gives $r = \dfrac{\pi - 1}{\pi}$.

VI. Give examples of the following:
(6)

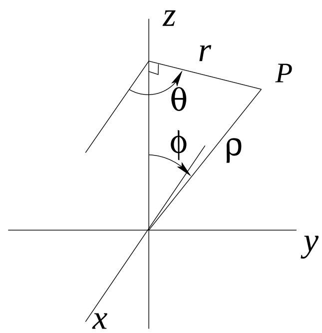
1. A power series $\sum_{n=0}^{\infty} c_n x^n$ that converges only for $x = 0$.

$\sum_{n=0}^{\infty} n! x^n$. For if we apply the Ratio Test, we obtain $\lim_{n\to\infty} \dfrac{(n+1)!\,|x^{n+1}|}{n!\,|x^n|} = \lim_{n\to\infty}(n+1)|x|$, which diverges to ∞ for any $x \neq 0$. So the series converges only for $x = 0$.

2. A power series $\sum_{n=0}^{\infty} c_n x^n$ whose radius of convergence is π.

Since $\sum_{n=0}^{\infty} x^n$ converges only for $-1 < x < 1$, the series $\sum_{n=0}^{\infty} \left(\dfrac{x}{\pi}\right)^n = \sum_{n=0}^{\infty} \left(\dfrac{1}{\pi}\right)^n x^n$ converges only for $-1 < \dfrac{x}{\pi} < 1$, that is, $-\pi < x < \pi$. Therefore it has radius of convergence π. [Alternatively, we could just take a series as in problem I with $b = \pi$, such as $\sum_{n=1}^{\infty} \dfrac{1}{n\pi^n} x^n$.]

VII. (3) Derive these formulas expressing rectangular coordinates in terms of spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.



From the right triangle shown in the figure, we read off $z = \rho\cos(\phi)$ and $r = \rho\sin(\phi)$. Then, using the formula for polar coordinates in the horizontal plane containing P, we have $x = \rho\sin(\phi)\cos(\theta)$ and $x = \rho\sin(\phi)\sin(\theta)$.

VIII. (4) In higher dimensions, say dimension n, there are vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ that play the roles of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$. In particular, $\vec{e}_i \cdot \vec{e}_j = 0$ when $i \neq j$, and $\vec{e}_i \cdot \vec{e}_i = 1$ for each i. Verify that if an n-dimesional vector $\vec{v}$ equals $r_1\vec{e_1} + r_2\vec{e_2} + \cdots + r_n\vec{e_n}$, then $r_i = \vec{v} \cdot \vec{e}_i$ for each i.

We calculate

$$\begin{aligned} \vec{v} \cdot \vec{e}_i &= (r_1\vec{e_1} + r_2\vec{e_2} + \cdots + r_i\vec{e} + \cdots + r_n\vec{e_n}) \cdot \vec{e}_1 \\ &= r_1\ \vec{e_1} \cdot \vec{e}_i + r_2\ \vec{e_2} \cdot \vec{e}_i + \cdots + r_i\vec{e_i} \cdot \vec{e}_i + \cdots + \cdot r_n\ \vec{e_n} \cdot \vec{e}_i \\ &= r_1 \cdot 0 + r_2 \cdots 0 + \cdots + r_i \cdot 1 + \cdots + r_n \cdot 0 = r_i\ . \end{aligned}$$

IX. (6) Give examples of the following:

1. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which $(\vec{a} \times \vec{b}\,) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$.

   $\vec{\imath} \times (\vec{\imath} \times \vec{\jmath}) = \vec{\imath} \times \vec{k} = -\vec{\jmath}$, but $(\vec{\imath} \times \vec{\imath}) \times \vec{\jmath} = \vec{0} \times \vec{\jmath} = \vec{0}$.

2. Nonzero vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ for which $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ but $\vec{b} \neq \vec{c}$.

   $\vec{\imath} \times (\vec{\imath} + \vec{\jmath}) = \vec{\imath} \times \vec{\imath} + \vec{\imath} \times \vec{\jmath} = \vec{\imath} \times \vec{\jmath}$, but $\vec{\imath} + \vec{\jmath} \neq \vec{\jmath}$.

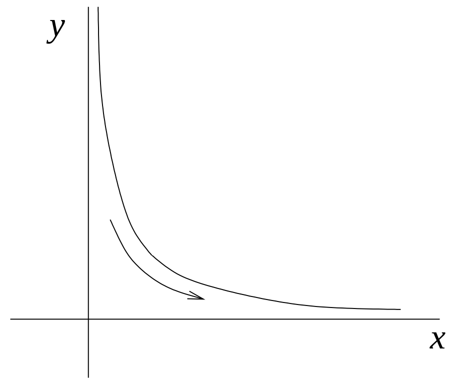
X. (5) Find an equation for the plane that contains the points $(1, 2, 3)$, $(1, 3, 4)$, and $(2, 3, 5)$.

Calling these points P, Q, and R, the vector from P to Q is $\vec{\jmath} + \vec{k}$ and the vector from Q to R is $\vec{\imath} + \vec{k}$. The cross product of these is normal to the plane that contains the points, and we compute it to be $\vec{\imath} + \vec{\jmath} - \vec{k}$. Since $(1, 2, 3)$ lies in the plane, an equation for the plane is $1(x-1) + 1(y-2) - 1(z-3) = 0$, or $x + y - z = 0$.

XI. A point moves according to the vector-valued function $\vec{r}(t) = e^t\vec{\imath} + e^{-t}\vec{\jmath}$.
(9)

1. Sketch the path of the point, indicating the direction of motion. (Hint: How are x and y related?)

   We observe that $y = \dfrac{1}{x}$ and $x > 0$ is increasing, giving the motion:



2. Calculate the velocity vectors $\vec{r}\,'(t)$, the speed, and the unit tangent vector $\vec{T}(t)$.

   The velocity vectors are $\vec{r}\,'(t) = e^t\,\vec{\imath} - e^{-t}\,\vec{\jmath}$, so the speed is $\|\, e^t\,\vec{\imath} - e^{-t}\,\vec{\jmath}\| = \sqrt{e^{2t} + e^{-2t}}$.

   The unit tangent vector is $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\,\vec{r}\,'(t)\,\|} = \dfrac{e^2\,\vec{\imath} - e^{-t}\,\vec{\jmath}}{\sqrt{e^{2t} + e^{-2t}}}$.

3. Use $a_T = \dfrac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\,\vec{r}\,'(t)\,\|}$ and $a_N = \dfrac{\|\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,\|}{\|\,\vec{r}\,'(t)\,\|}$ to calculate the tangential and normal components of the acceleration vector $\vec{a}(t)$.

   First we compute $\vec{r}\,''(t) = e^t\,\vec{\imath} + e^{-t}\,\vec{\jmath}$, that is, $\vec{r}\,''(t) = \vec{r}(t)$. So we have

$$a_T = \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\|\,\vec{r}\,'(t)\,\|} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}\ ,$$

   and

$$a_N = \frac{\|\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,\|}{\|\,\vec{r}\,'(t)\,\|} = \frac{\|\,2\,\vec{k}\,\|}{\sqrt{e^{2t} + e^{-2t}}} = \frac{2}{\sqrt{e^{2t} + e^{-2t}}}\ .$$

4. When is the point speeding up? When is it slowing down?

   It is speeding up when the tangential component a_T is positive, that is, when $e^{2t} > e^{-2t}$. Applying logarithm gives $2t > -2t$ or $t > 0$. Similarly, it is slowing down when the tangential component is negative, which is when $t < 0$.

XII. Write the general formula for the Taylor series of a function $f(x)$ at $x = a$. Use it to calculate the Taylor
(6) series of the function $f(x) = x^4$ at $x = 2$.

The general form is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

$f(2) = 16$, $f'(2) = 4\cdot 2^3 = 32$, $f''(2) = 4\cdot 3\cdot 2^2 = 48$, $f^{(3)}(2) = 4\cdot 3\cdot 2\cdot 2^1 = 48$, $f^{(4)}(2) = 4\cdot 3\cdot 2\cdot 1 = 24$, and all higher derivatives are 0. So the Taylor series is

$$f(2) + f'(2)(x-2) + (f''(2)/2!)(x-2)^2 + (f^{(3)}(2)/3!)(x-2)^3 + (f^{(4)}(2)/4!)(x-2)^4$$
$$= 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4\ .$$

XIII. For the helix $\vec{r}(t) = 2\sin(t)\vec{\imath} + 3t\vec{\jmath} + 2\cos(t)\vec{k}$:
(8)

1. Calculate the unit tangent vector $\vec{T}(t)$, and use it to calculate the unit normal $\vec{N}(t)$.

   Regarded as a vector-valued function of t, the helix is $\vec{r}(t) = 2\sin(t)\vec{\imath} + 3t\vec{\jmath} + 2\cos(t)\vec{k}$, and

$$\vec{v}(t) = \vec{r}\,'(t) = 2\cos(t)\vec{\imath} + 3\vec{\jmath} - 2\sin(t)\vec{k}$$
$$\|\, v(t)\,\| = \sqrt{4\cos^2(t) + 9 + 4\sin^2(t)} = \sqrt{13}$$
$$\vec{T}(t) = \vec{v}(t)/\|\,\vec{v}(t)\,\| = \frac{2\cos(t)}{\sqrt{13}}\,\vec{\imath} + \frac{3}{\sqrt{13}}\,\vec{\jmath} - \frac{2\sin(t)}{\sqrt{13}}\,\vec{k}\ .$$

   A normal vector is $\vec{T}'(t) = \dfrac{-2\sin(t)}{\sqrt{13}}\vec{\imath} - \dfrac{2\cos(t)}{\sqrt{13}}\vec{k}$. Since $\|\,\vec{T}'(t)\,\| = \sqrt{4\sin^2(t)/13 + 4\cos^2(t)/13} = \dfrac{2}{\sqrt{13}}$, the unit normal is $\vec{N}(t) = -\sin(t)\vec{\imath} - \cos(t)\vec{k}$.

2. Use the formula $\kappa = \dfrac{\|\,\vec{T}\,'(t)\,\|}{\|\,\vec{r}\,'(t)\,\|}$ to calculate the curvature.

$$\kappa = \frac{\|\,\vec{T}\,'(t)\,\|}{\|\,\vec{r}\,'(t)\,\|} = \frac{2/\sqrt{13}}{\sqrt{13}} = \frac{2}{13}.$$

3. Use the formula $\kappa = \left\|\dfrac{d\vec{T}}{ds}\right\|$ and the Chain Rule to calculate the curvature.

   We have $\dfrac{ds}{dt} = \|\,\vec{v}(t)\,\| = \sqrt{13}$, so

$$\kappa = \left\|\,\frac{d\vec{T}}{ds}\,\right\| = \left\|\,\frac{d\vec{T}}{dt}\Big/\frac{ds}{dt}\,\right\| = \frac{1}{\sqrt{13}}\left\|\,\frac{d\vec{T}}{dt}\,\right\| = \frac{1}{\sqrt{13}}\,\frac{2}{\sqrt{13}} = \frac{2}{13}\ .$$

XIV. Bonus Problem: Let $u = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots$, $v = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots$, and $w = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots$.
(6) Each of these converges by comparison with the Maclaurin Series of e^x. Show that $u^3 + v^3 + w^3 - 3uvw = 1$. (Hint: What is u'?)

We compute that $u' = 0 + \dfrac{3x^2}{3!} + \dfrac{6x^5}{6!} + \dfrac{9x^8}{9!} + \cdots = 0 + \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots = w$, and similarly $v' = u$ and $w' = v$. Then we have

$$\begin{aligned}(u^3 + v^3 + w^3 - 3uvw)' &= 3u^2u' + 3v^2v' + 3w^2w' - 3u'vw - 3uv'w - 3uvw' \\ &= 3u^2w + 3v^2u + 3w^2v - 3vw^2 - 3wu^2 - 3uv^2 = 0\ .\end{aligned}$$

This implies that $u^3 + v^3 + w^3 - 3uvw$ equals some constant. Evaluating $u^3 + v^3 + w^3 - 3uvw$ when $x = 0$ gives $1^3 + 0 + 0 - 3 \cdot 1 \cdot 0 \cdot 0 = 1$, so the constant equals 1. That is, $u^3 + v^3 + w^3 - 3uvw = 1$.