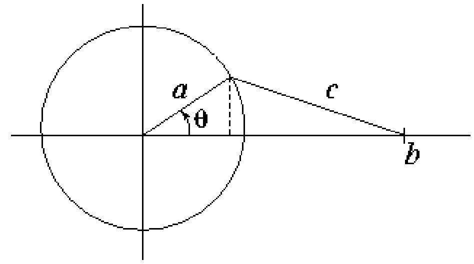
Mathematics 1823-001H Name (please print) ____________________

Examination III

November 21, 2006

Instructions: Give brief, clear answers. It is not expected that most people will be able to answer all the questions, just do what you can in 75 minutes.

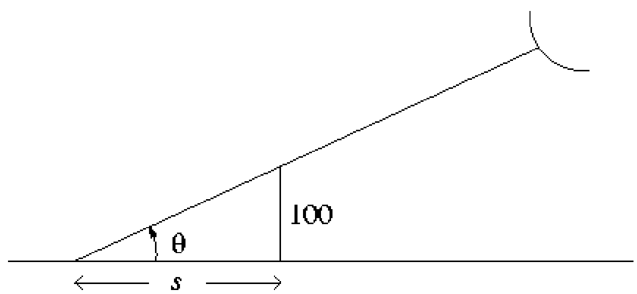
I. (6) State the Law of Cosines, and verify it. A helpful figure is shown to the right.



The Law of Cosines says that if a, b, and c are the sides of a triangle, and θ is the angle where the sides of lengths a and b meet, then $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

To verify the Law of Cosines, we first note that the coordinate of the vertex of the triangle that lies on the circle is $(a\cos(\theta), a\sin(\theta))$. Therefore the height of the triangle is $a\sin(\theta)$. The base of the right-hand right triangle is $b - a\cos(\theta)$. Applying the Pythagorean Theorem to that triangle, we have $c^2 = a^2\sin^2(\theta) + (b - a\cos(\theta))^2 = a^2\sin^2(\theta) + b^2 - 2ab\cos(\theta) + a^2\cos^2(\theta) = a^2 + b^2 - 2ab\cos(\theta)$.

II. (6) The angle of elevation of the sun is decreasing at 0.25 radians per hour. How fast is the length of the shadow of a 100 meter tall tower changing at a time (around 4 p. m.) when the angle of elevation of the sun is $\pi/6$?



We are given that $\dfrac{d\theta}{dt} = -0.25$, and the problem asks for $\left.\dfrac{ds}{dt}\right|_{\theta=\pi/6}$. From the above diagram, we have $s/100 = \cot(\theta)$. Differentiating gives $\dfrac{1}{100}\dfrac{ds}{dt} = -\csc^2(\theta)\dfrac{d\theta}{dt}$, so $\dfrac{ds}{dt} = -100\csc^2(\theta)\dfrac{d\theta}{dt}$. Evaluating when $\theta = \pi/6$, we have $\left.\dfrac{ds}{dt}\right|_{\theta=\pi/6} = -100\csc^2(\pi/6)\left.\dfrac{d\theta}{dt}\right|_{\theta=\pi/6} = -100 \cdot 2^2 \cdot (-0.25) = 100$. That is, the length of the shadow is increasing at 100 meters per hour.

III. (15) The Mean Value Theorem states that if a function f is differentiable at all points between a and b, and is continuous at a and b as well, then there exists a c between a and b so that $f(b) - f(a) = f'(c)\,(b - a)$.

1. Find a value that works as the number c in the Mean Value Theorem for the function $x^{2/3}$ on the interval $[0, 8]$.

   We need $8^{2/3} - 0^{2/3} = f'(c)(8-0)$. Since $f'(x) = 2x^{-1/3}/3$, this says that $4 = 8 \cdot 2c^{-1/3}/3$. So $c^{1/3} = 4/3$, giving $c = 64/27$.

2. Verify that if $f'(x) \leq 0$ for all x with $a \leq x \leq b$, then $f(b) \leq f(a)$.

   $f(b) - f(a) = f'(c)(b-a) \leq 0$, the latter inequality since $f'(c) \leq 0$ and $b - a > 0$. So we have $f(b) \geq f(a)$.

3. Verify that if $f'(x) = 0$ for all x in a (connected, but not necessarily closed) interval, then f is constant on the interval.

   Choose some point x_0 in the interval, and let x be any other point in the interval. Applying the Mean Value Theorem, we have $f(x) - f(x_0) = f'(c)(x - x_0)$. Since c is between x and x_0, it must also lie in the interval, and $f'(c) = 0$. So $f(x) = f(x_0)$. This is true for all x in the interval, so f is constant.

4. Show that the function $2x - 3 - \sin(x)$ has at most one root between -5 and 5.

   Letting $f(x)$ be this function, we have $f'(x) = 2 - \cos(x)$. Since $-1 \leq \cos(x) \leq 1$, the derivative is nonzero at all points. If the function had two roots r_1 and r_2 in the interval, then by the Mean Value Theorem we would have $0 = f(r_1) - f(r_2) = f'(c)(r_1 - r_2)$ for some c between r_1 and r_2, giving $f'(c) = 0$. This is impossible since $f'(x)$ is never equal to 0.

5. Show that the function $2x - 3 - \sin(x)$ has at least one root between -5 and 5.

   Letting $f(x)$ be this function, we have $f(-5) = -13 - \sin(-5) \leq -12$, and $f(5) = 7 - \sin(5) \geq 5$. Since f is continuous and $f(-5) < 0 < f(5)$, the Intermediate Value Theorem guarantees that there is a c between -5 and 5 so that $f(c) = 0$.

IV. (4) Find all critical points of the function $5t^{2/3} + t^{5/3}$.

The derivative is $10t^{-1/3}/3 + 5t^{2/3}/3$. This is undefined at $t = 0$, so $t = 0$ is one critical point. For nonzero values of t, we must solve $10t^{-1/3}/3 + 5t^{2/3}/3 = 0$. Factoring, we have $(5t^{-1/3}/3)\,(2 + t)$. Since $t^{-1/3}$ is never 0, the only other critical point is $t = -2$.

V. (4) One of the lines that passes through the point $(2, 0)$ and is tangent to the graph of $y = x^4$ is $y = 0$. Find the other one.

Let (x_0, x_0^4) be the point of tangency. The slope of the tangent line can be expressed either as $4x^3|_{x=x_0} = 4x_0^3$ or as $\dfrac{x_0^4 - 0}{x_0 - 2}$. Equating these and solving gives $x_0 = \frac{8}{3}$, so the slope is $4 \cdot (\frac{8}{3})^3 = \frac{2^{11}}{3^3}$. Since the line passes through $(2, 0)$, an equation is $y = \frac{2^{11}}{3^3}(x - 2)$.

VI. (4) The Extreme Value Theorem says that a continuous function on a closed interval must assume maximum and minimum values.

1. Give an example of a trigonometric function which is continuous on an open interval, and assumes neither a maximum nor a minimum value on the interval.

   $\tan(x)$ on the interval $(-\pi/2, \pi/2)$, for instance

2. Give an example of a trigonometric function which is continuous on an open interval, and assumes both maximum and minimum values on the interval.

   $\sin(x)$ on the interval $(0, 2\pi)$, for instance

VII. (12) A certain function $f(x)$ has derivative $f'(x) = \dfrac{x}{x^2+1}$.

1. Determine where $f'(x)$ is positive, and where it is negative.

   Since $x^2 + 1$ is always positive, $f'(x)$ is negative when $x < 0$ and positive when $x > 0$.

2. Calculate $f''(x)$. Determine where it is positive, and where it is negative.

   $f''(x) = \dfrac{1-x^2}{(x^2+1)^2}$. Since $(x^2+1)^2 > 0$ for all x, $f''(x)$ is negative when $1 - x^2 < 0$, i. e. when $x < 1$ or $x > 1$, and positive when $1 - x^2 > 0$, i. e. when $-1 < x < 1$.

3. Where does the minimum value of $f(x)$ occur? Why?

   Since $f'(x)$ changes from negative to positive only at $x = 0$, the minimum value of f must occur at $x = 0$.

4. Determine where $f(x)$ is concave up, and where it is concave down.

   $f(x)$ is concave up where $f''(x) > 0$, i. e. for $-1 < x < 1$, and is concave down where $f''(x) < 0$, i. e. for $x < -1$ and $x > 1$.

5. Find all inflection points of f.

   $f''(x)$ changes sign at $x = -1$ and $x = 1$, so the inflection points of f are at $x = \pm 1$.

VIII. (5) Use the definition of rate of change to show that if $f'(a) > 0$, then there exists a $\delta > 0$ so that if $a < a + h < a + \delta$, then $f(a) < f(x)$. Hint: Write $f(x) = f(a) + f'(a)h + E(h)$, where $\lim_{h\to 0} \dfrac{E(h)}{h} = 0$, and use the observation that $f(a) + f'(a)\,h + E(h) = f(a) + \left(f'(a) + \dfrac{E(h)}{h}\right) h$.

We have $\lim_{h\to 0} \dfrac{E(h)}{h} = 0$. Taking $\epsilon = f'(a)$ in the definition of limit, there exists a $\delta > 0$ so that if $0 < |h| < \delta$, then $\left|\dfrac{E(h)}{h}\right| < f'(a)$. This says that $-f'(a) < \dfrac{E(h)}{h} < f'(a)$, so $0 < f'(a) + \dfrac{E(h)}{h}$. In particular, when $a < a + h < a + \delta$, we have $0 < h < \delta$ so

$$f(x) = f(a) + f'(a)h + E(h) = f(a) + \left(f'(a) + \frac{E(h)}{h}\right) h > f(a)\ .$$