I. State the Law of Cosines, and verify it. A helpful figure is shown to the right.

The Law of Cosines says that if $a$, $b$, and $c$ are the sides of a triangle, and $\theta$ is the angle where the sides of lengths $a$ and $b$ meet, then

\[ c^2 = a^2 + b^2 - 2ab\cos(\theta). \]

To verify the Law of Cosines, we first note that the coordinate of the vertex of the triangle that lies on the circle is $(a\cos(\theta), a\sin(\theta))$. Therefore the height of the triangle is $a\sin(\theta)$. The base of the right-hand right triangle is $b - a\cos(\theta)$. Applying the Pythagorean Theorem to that triangle, we have

\[ c^2 = a^2\sin^2(\theta) + (b - a\cos(\theta))^2 = a^2\sin^2(\theta) + b^2 - 2ab\cos(\theta) + a^2\cos^2(\theta) = a^2 + b^2 - 2ab\cos(\theta). \]

II. The angle of elevation of the sun is decreasing at 0.25 radians per hour. How fast is the length of the shadow of a 100 meter tall tower changing at a time (around 4 p.m.) when the angle of elevation of the sun is $\pi/6$?

We are given that $\frac{d\theta}{dt} = -0.25$, and the problem asks for $\left. \frac{ds}{dt}\right|_{\theta=\pi/6}$. From the above diagram, we have

\[ s/100 = \cot(\theta). \]

Differentiating gives

\[ \frac{1}{100} \frac{ds}{dt} = -\csc^2(\theta) \frac{d\theta}{dt}, \]

so $\frac{ds}{dt} = -100\csc^2(\theta) \frac{d\theta}{dt}$. Evaluating when $\theta = \pi/6$, we have

\[ \left. \frac{ds}{dt}\right|_{\theta=\pi/6} = -100\csc^2(\pi/6) \left. \frac{d\theta}{dt}\right|_{\theta=\pi/6} = -100 \cdot 2^2 \cdot (-0.25) = 100. \]

That is, the length of the shadow is increasing at 100 meters per hour.

III. The Mean Value Theorem states that if a function $f$ is differentiable at all points between $a$ and $b$, and is continuous at $a$ and $b$ as well, then there exists a $c$ between $a$ and $b$ so that $f(b) - f(a) = f'(c)(b - a)$.

1. Find a value that works as the number $c$ in the Mean Value Theorem for the function $x^{2/3}$ on the interval $[0, 8]$.

We need $8^{2/3} - 0^{2/3} = f'(c)(8 - 0)$. Since $f'(x) = 2x^{-1/3}/3$, this says that $4 = 8 \cdot 2c^{-1/3}/3$. So $c^{1/3} = 4/3$, giving $c = 64/27$.

2. Verify that if $f'(x) \leq 0$ for all $x$ with $a \leq x \leq b$, then $f(b) \leq f(a)$.

\[ f(b) - f(a) = f'(c)(b - a) \leq 0, \] the latter inequality since $f'(c) \leq 0$ and $b - a > 0$. So we have $f(b) \geq f(a)$. 

3. Verify that if \( f'(x) = 0 \) for all \( x \) in a (connected, but not necessarily closed) interval, then \( f \) is constant on the interval.

Choose some point \( x_0 \) in the interval, and let \( x \) be any other point in the interval. Applying the Mean Value Theorem, we have \( f(x) - f(x_0) = f'(c)(x - x_0) \). Since \( c \) is between \( x \) and \( x_0 \), it must also lie in the interval, and \( f'(c) = 0 \). So \( f(x) = f(x_0) \). This is true for all \( x \) in the interval, so \( f \) is constant.

4. Show that the function \( 2x - 3 - \sin(x) \) has at most one root between \(-5\) and \(5\).

Letting \( f(x) \) be this function, we have \( f'(x) = 2 - \cos(x) \). Since \(-1 \leq \cos(x) \leq 1\), the derivative is nonzero at all points. If the function had two roots \( r_1 \) and \( r_2 \) in the interval, then by the Mean Value Theorem we would have \( 0 = f(r_1) - f(r_2) = f'(c)(r_1 - r_2) \) for some \( c \) between \( r_1 \) and \( r_2 \), giving \( f'(c) = 0 \). This is impossible since \( f'(x) \) is never equal to 0.

5. Show that the function \( 2x - 3 - \sin(x) \) has at least one root between \(-5\) and \(5\).

Letting \( f(x) \) be this function, we have \( f(-5) = -13 - \sin(-5) \leq -12 \), and \( f(5) = 7 - \sin(5) \geq 5 \). Since \( f \) is continuous and \( f(-5) < 0 < f(5) \), the Intermediate Value Theorem guarantees that there is a \( c \) between \(-5\) and \(5\) so that \( f(c) = 0 \).

IV. Find all critical points of the function \( 5t^{2/3} + t^{5/3} \).

The derivative is \( 10t^{-1/3}/3 + 5t^{2/3}/3 \). This is undefined at \( t = 0 \), so \( t = 0 \) is one critical point. For nonzero values of \( t \), we must solve \( 10t^{-1/3}/3 + 5t^{2/3}/3 = 0 \). Factoring, we have \( (5t^{-1/3}/3)(2 + t) \). Since \( t^{-1/3} \) is never 0, the only other critical point is \( t = -2 \).

V. One of the lines that passes through the point \((2, 0)\) and is tangent to the graph of \( y = x^4 \) is \( y = 0 \). Find the other one.

Let \( (x_0, x_0^4) \) be the point of tangency. The slope of the tangent line can be expressed either as \( 4x^3 \big|_{x=x_0} = 4x_0^3 \) or as \( \frac{x_0^4 - 0}{x_0 - 2} \). Equating these and solving gives \( x_0 = \frac{8}{3} \), so the slope is \( 4 \cdot \left(\frac{8}{3}\right)^3 = \frac{211}{37} \). Since the line passes through \((2, 0)\), an equation is \( y = \frac{211}{37}(x - 2) \).

VI. The Extreme Value Theorem says that a continuous function on a closed interval must assume maximum and minimum values.

1. Give an example of a trigonometric function which is continuous on an open interval, and assumes neither a maximum nor a minimum value on the interval.

\( \tan(x) \) on the interval \((-\pi/2, \pi/2)\), for instance

2. Give an example of a trigonometric function which is continuous on an open interval, and assumes both maximum and minimum values on the interval.

\( \sin(x) \) on the interval \((0, 2\pi)\), for instance
A certain function \( f(x) \) has derivative \( f'(x) = \frac{x}{x^2 + 1} \).

1. Determine where \( f'(x) \) is positive, and where it is negative.

   Since \( x^2 + 1 \) is always positive, \( f'(x) \) is negative when \( x < 0 \) and positive when \( x > 0 \).

2. Calculate \( f''(x) \). Determine where it is positive, and where it is negative.

   \[ f''(x) = \frac{1 - x^2}{(x^2 + 1)^2} \]

   Since \( (x^2 + 1)^2 > 0 \) for all \( x \), \( f''(x) \) is negative when \( 1 - x^2 < 0 \), i.e. when \( x < 1 \) or \( x > 1 \), and positive when \( 1 - x^2 > 0 \), i.e. when \( -1 < x < 1 \).

3. Where does the minimum value of \( f(x) \) occur? Why?

   Since \( f'(x) \) changes from negative to positive only at \( x = 0 \), the minimum value of \( f \) must occur at \( x = 0 \).

4. Determine where \( f(x) \) is concave up, and where it is concave down.

   \( f(x) \) is concave up where \( f''(x) > 0 \), i.e. for \(-1 < x < 1\), and is concave down where \( f''(x) < 0 \), i.e. for \( x < -1 \) and \( x > 1 \).

5. Find all inflection points of \( f \).

   \( f''(x) \) changes sign at \( x = -1 \) and \( x = 1 \), so the inflection points of \( f \) are at \( x = \pm 1 \).

Use the definition of rate of change to show that if \( f'(a) > 0 \), then there exists a \( \delta > 0 \) so that if \( a < a + h < a + \delta \), then \( f(a) < f(x) \).

Hint: Write \( f(x) = f(a) + f'(a)h + E(h) \), where \( \lim_{h \to 0} \frac{E(h)}{h} = 0 \), and use the observation that \( f(a) + f'(a)h + E(h) = f(a) + f'(a) + \frac{E(h)}{h} \) h.

We have \( \lim_{h \to 0} \frac{E(h)}{h} = 0 \). Taking \( \epsilon = f'(a) \) in the definition of limit, there exists a \( \delta > 0 \) so that if \( 0 < |h| < \delta \), then \( \left| \frac{E(h)}{h} \right| < f'(a) \). This says that \( -f'(a) < \frac{E(h)}{h} < f'(a) \), so \( 0 < f'(a) + \frac{E(h)}{h} \). In particular, when \( a < a + h < a + \delta \), we have \( 0 < h < \delta \) so

\[ f(x) = f(a) + f'(a)h + E(h) = f(a) + f'(a) + \frac{E(h)}{h} h > f(a). \]