Mathematics 1823-001H Name (please print) ______________

Examination II

October 26, 2006

Instructions: Give brief, clear answers. It is not expected that most people will be able to answer all the questions, just do what you can in 75 minutes.

I. (12) Find the derivatives of the following functions, using the specified methods.

1. $\sqrt{1+2x}$, using the definition $\lim\limits_{h\to x} \dfrac{f(h)-f(x)}{h-x}$.

$$\begin{aligned}\lim_{h\to x} \frac{f(h)-f(x)}{h-x} &= \lim_{h\to x} \frac{\sqrt{1+2h}-\sqrt{1+2x}}{h-x} = \lim_{h\to x} \frac{\sqrt{1+2h}-\sqrt{1+2x}}{h-x} \cdot \frac{\sqrt{1+2h}+\sqrt{1+2x}}{\sqrt{1+2h}+\sqrt{1+2x}} \\ &= \lim_{h\to x} \frac{1+2h-(1+2x)}{h-x} \cdot \frac{1}{\sqrt{1+2h}+\sqrt{1+2x}} = \lim_{h\to x} \frac{2(h-x)}{h-x} \cdot \frac{1}{\sqrt{1+2h}+\sqrt{1+2x}} \\ &= 2 \cdot \frac{1}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}\end{aligned}$$

2. $1/x^2$, using the definition $\lim\limits_{w\to 0} \dfrac{f(x+w)-f(x)}{w}$.

$$\lim_{w\to 0} \frac{f(x+w)-f(x)}{w} = \lim_{w\to 0} \frac{\dfrac{1}{(x+w)^2} - \dfrac{1}{x^2}}{w} = \lim_{w\to 0} \frac{-2xw-w^2}{wx^2(x+w)^2} = \lim_{w\to 0} \frac{-2x-w}{x^2(x+w)^2} = -\frac{2x}{x^2(x+0)^2} = -\frac{2}{x^3}$$

3. x^3, using the definition as the best linear approximation.

$(x+h)^3 = x^3+3x^2h+3xh^2+h^3 = x^3+(3x^2)\cdot h+(3xh^2+h^3)$. Since $\lim\limits_{h\to 0} \dfrac{3xh^2+h^3}{h} = \lim\limits_{h\to 0} 3xh+h^2 = 3x\cdot 0+0^2 = 0$, this shows that the rate of change of x^3 is $3x^2$.

4. A function that stretches the line segment $[1/3, 2]$ uniformly to the line segment $[-2/3, 2/3]$, and also reverses its direction, using the concept of derivative as stretch factor.

The first interval has length $5/3$ and and the second has length $4/3$, so the interval was uniformly stretched by a factor of $4/5$ (that is, was compressed to $4/5$ its original length). Since the direction was reversed, the derivative at each point is $-4/5$.

II. (9) Find the derivatives of the following functions, using the algebraic rules for sums, products, quotients, and the Chain Rule, and the known derivatives of x^r and the trigonometric functions.

1. $\dfrac{ax+b}{cx+d}$ (where a, b, c, and d are constants).

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{(cx+d)a-(ax+b)c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$$

2. $\csc^3(x)\sqrt{\sin(x)}$

$$\begin{aligned}(\csc^3(x)\sqrt{\sin(x)})' &= 3\csc^2(x)\cdot(-\csc(x)\cot(x))\sqrt{\sin(x)} + \csc^3(x)\cdot(1/2)(\sin(x))^{-1/2}\cdot\cos(x) \\ &= \frac{\csc^2(x)(-3\cot(x)+\csc(x)\cos(x)/2)}{\sqrt{\sin(x)}} = \frac{-(5/2)\csc^2(x)\cot(x)}{\sqrt{\sin(x)}}\end{aligned}$$

3. $L(1/x)$, where $L(x)$ is a function such that $L'(x) = 1/x$.

$$\frac{d}{dx}(L(1/x)) = L'(1/x) \cdot \frac{d}{dx}(1/x) = \frac{1}{1/x} \cdot (-1/x^2) = -1/x$$

III. (5) For the function $f(x) = \sqrt{x}$, obtain a general formula for the n^{th} derivative $f^{(n)}(x)$.

$f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = \frac{1}{2}\frac{-1}{2}x^{-3/2}$, $f^{(3)}(x) = \frac{1}{2}\frac{(-1)}{2}\frac{(-3)}{2}x^{-5/2}$, $f^{(4)}(x) = \frac{1}{2}\frac{(-1)}{2}\frac{(-3)}{2}\frac{(-5)}{2}x^{-7/2}$, $f^{(5)}(x) = \frac{1}{2}\frac{(-1)}{2}\frac{(-3)}{2}\frac{(-5)}{2}\frac{(-7)}{2}x^{-9/2}$, $f^{(6)}(x) = \frac{1}{2}\frac{(-1)}{2}\frac{(-3)}{2}\frac{(-5)}{2}\frac{(-7)}{2}\frac{(-9)}{2}x^{-11/2}$. We can write this as $f^{(6)}(x) = \frac{(-1)^5}{2^6}1 \cdot 3 \cdot 5 \cdot 7 \cdot 9\, x^{-11/2}$. Another differentiation gives $f^{(7)}(x) = \frac{(-1)^6}{2^7}1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11\, x^{-13/2}$. The general pattern is $f^{(n)}(x) = \frac{(-1)^{n-1}}{2^n}1 \cdot 3 \cdot 5 \cdots (2n-3)\, x^{-(2n-1)/2}$.

IV. (4) Calculate the following limits.

1. $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta^2}$

   $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta^2} = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \frac{1}{\theta}$. Since $\frac{\sin(\theta)}{\theta}$ is close to 1 for θ near 0, the function $\frac{\sin(\theta)}{\theta} \cdot \frac{1}{\theta}$ will behave like $\frac{1}{\theta}$, so the limit does not exist.

2. $\lim_{\theta \to 0} \frac{\sin(8\theta)}{\sin(7\theta)}$

   $$\lim_{\theta \to 0} \frac{\sin(8\theta)}{\sin(7\theta)} = \lim_{\theta \to 0} \frac{\sin(8\theta)}{8\theta} \frac{7\theta}{\sin(7\theta)} \frac{8}{7} = 1 \cdot 1 \cdot \frac{8}{7} = \frac{8}{7}$$

3. $\lim_{\theta \to 0} \frac{\sin^2(\theta)}{\theta}$

   $$\lim_{\theta \to 0} \frac{\sin^2(\theta)}{\theta} = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \sin(\theta) = 1 \cdot 0 = 0$$

4. $\lim_{\theta \to 0} \frac{\sin(\theta^2)}{\theta^2}$

   $\lim_{\theta \to 0} \frac{\sin(\theta^2)}{\theta^2} = 1$, since θ^2 is close to 0 when x is close to 0.

V. (3) Calculate $\frac{dy}{dx}$ using implicit differentiation if $xy^2 = \cot(xy)$.

$$\begin{aligned} x\,2y\frac{dy}{dx} + y^2 &= -\csc^2(xy) \cdot (y + x\frac{dy}{dx}) \\ (2xy + x\csc^2(xy))\frac{dy}{dx} &= -y\csc^2(xy) - y^2 \\ \frac{dy}{dx} &= -\frac{y\csc^2(xy) + y^2}{2xy + x\csc^2(xy)} \end{aligned}$$

VI. (3) Calculate $\dfrac{d}{dx}(f(g(x)))\Big|_{x=1}$ if f and g are as given in the table:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

$$\frac{d}{dx}(f(g(x)))\Big|_{x=1} = f'(g(x))\, g'(x)\Big|_{x=1} = f'(g(1))\, g'(1) = f'(2)\, 6 = 5 \cdot 6 = 30$$

VII. (4) One of the lines that passes through the point $(1,0)$ and is tangent to the graph of $y = x^3$ is $y = 0$. Find the other one.

Let (x_0, x_0^3) be the point of tangency. The slope of the tangent line can be expressed either as $3x^2|_{x=x_0} = 3x_0^2$ or as $\dfrac{x_0^3 - 0}{x_0 - 1}$. Equating these and solving gives $x_0 = \frac{3}{2}$, so the slope is $3 \cdot (\frac{3}{2})^2 = \frac{27}{4}$. Since the line passes through $(1,0)$, an equation is $y = \frac{27}{4}(x-1)$.

VIII. (3) Use $1 = f \cdot (1/f)$ and the product rule to obtain a formula for $(1/f)'$.

Differntiating both sides of $1 = f \cdot (1/f)$ gives $0 = f' \cdot (1/f) + f \cdot (1/f)'$. Solving gives $(1/f)' = -f'/f^2$.

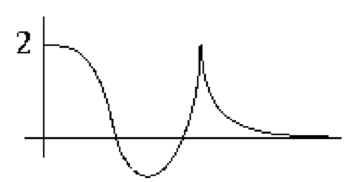
IX. (3) Verify that if f is an odd function, then f' is an even function.

If f is odd, then we have

$$\begin{aligned} f(-x) &= -f(x) \\ f'(-x) \cdot (-1) &= -f'(x) \\ f'(-x) &= f'(x) \end{aligned}$$

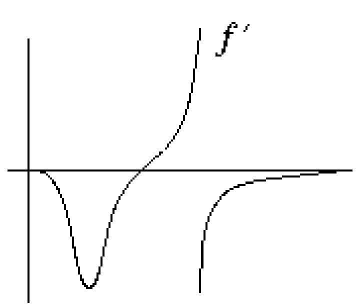
so f' is even.

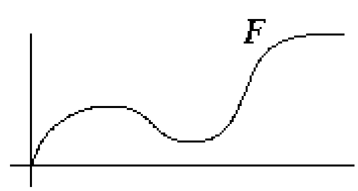
X. (6) To the right is the graph of a function f. On two separate graphs:



(a) Sketch f'.

(b) Sketch a function F for which $F(0) = 0$ and $F' = f$.





XI. (3) Challenge Problem: Calculate $\lim_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$ by using one of the definitions of the derivative.

$\lim_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1} \sqrt{x}\,\dfrac{1-(\sqrt{x})^3}{1-\sqrt{x}} = \lim_{x\to 1} \sqrt{x}\,\dfrac{(\sqrt{x})^3-1}{\sqrt{x}-1}$. If we put $f(x)=x^3$, then $\lim_{x\to 1} \dfrac{(\sqrt{x})^3-1}{\sqrt{x}-1} = \lim_{x\to 1} \dfrac{f(\sqrt{x})-f(1)}{\sqrt{x}-1}$. Since $\sqrt{x}$ approaches 1 as x approaches 1, the latter is the same as $\lim_{x\to 1} \dfrac{f(x)-f(1)}{x-1}$. This is one of the definitions of $f'(1)$. Since $f'(x)=3x^2$, $f'(1)$ equals 3. So $\lim_{x\to 1} \sqrt{x}\,\dfrac{(\sqrt{x})^3-1}{\sqrt{x}-1} = 1\cdot 3 = 3$ (just as in problem VI.1 of Exam I).