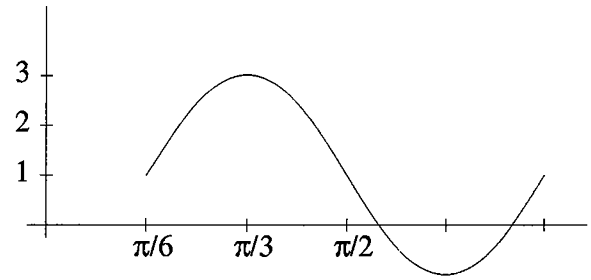
Mathematics 1823-001H Name (please print) ____________________

Examination I

September 26, 2006

Instructions: Give brief, clear answers. It is not expected that most people will be able to answer all the questions, just do what you can in 75 minutes.

I. (4) The figure to the right shows the graph of a certain function which is obtained from the standard sine function by vertical and horizontal translation and stretching. Determine an expression of the form $y = A\sin(Bx + C) + D$ for this graph.



Start with the standard graph $y = \sin(x)$. The period of the illustrated function is $2\pi/3$, so we change $\sin(x)$ to $\sin(3x)$ to stretch horizontally by a factor of $1/3$. The graph that we want starts at $x = \pi/6$ rather than $x = 0$, so we change $\sin(3x)$ to $\sin(3(x - \pi/6)) = \sin(3x - \pi/2)$. Finally, make the amplitude 2 by changing to $2\sin(3x - \pi/2)$, and translate upward vertically by 1 to obtain $y = 2\sin(3x - \pi/2) + 1$. (These steps can be done in different orders, but will arrive at the same expression.)

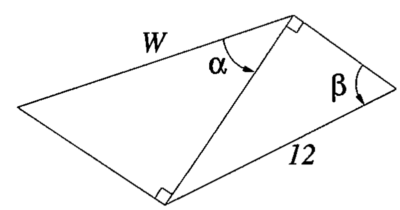
II. (6) Sketch a graph of the function $\cos(1/x)$. On another coordinate system, sketch a graph of the function $x^2\cos(1/x)$. State the Squeeze Theorem, and explain how it applies to find $\lim_{x\to 0} x^2\cos(1/x)$.

For the graphs, see the last page. The Squeeze Theorem says that if $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x) \le g(x) \le h(x)$ for all x near a, but not necessarily at $x = a$, and such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists and equals L. For the functions $f(x) = -x^2$, $g(x) = x^2\cos(1/x)$, and $h(x) = x^2$, we have $-1 \le \cos(1/x) \le 1$, so $-x^2 \le x^2\cos(1/x) \le x^2$. Since $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} -x^2 = 0$, the Squeeze Theorem ensures that $\lim_{x\to 0} x^2\cos(1/x) = 0$

III. (2) Give a precise formal definition of $\lim_{\delta\to L} G(\delta) = x$.

For every $\epsilon > 0$, there exists $\omega > 0$ so that if δ satisfies $0 < |\delta - L| < \omega$, then $|G(\delta) - x| < \epsilon$.

IV. (4) The figure to the right shows two right triangles, with two angles labeled α and β, and a side whose length is shown to be 12. Find an expression involving α and β for the length of the side labeled as W.



The side of the right-hand triangle opposite the angle β has length $12\sin(\beta)$, so the side W has length $12\sin(\beta)\sec(\alpha)$.

V. (4) A rectangular box with volume 7 m^3 has square base and open top. Find the height $h(x)$ of the box and the length $\ell(x)$ of a diagonal of one of its sides as a function of the length x of a side of the base.

The volume is $7 = x^2h(x)$ so $h(x) = 7/x^2$. The sides of the box are rectangles with one side x and the other side $h(x)$, so the Pythagorean Theorem gives $\ell(x) = \sqrt{x^2 + (7/x^2)^2} = \sqrt{x^6 + 49}/x^2$.

VI. (12) Calculate the following limits. Make use of the fact that $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$, when necessary. Give enough explanation to make it clear that you understand where your answer is coming from. Do not use l'Hôpital's Rule.

1. $\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}}$

$$\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1} \sqrt{x}\cdot\frac{1-(\sqrt{x})^3}{1-\sqrt{x}} = \lim_{x\to 1} \sqrt{x}\cdot\frac{1-\sqrt{x}}{1-\sqrt{x}}\cdot(1+\sqrt{x}+x) = 1\cdot 1\cdot(1+1+1) = 3.$$

2. $\lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta^2}$

$$\lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta^2} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{\theta} = 1\cdot 1 = 1$$

3. $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta^2}$

$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta^2} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\cdot\frac{1}{\theta}$. Since $\frac{\sin(\theta)}{\theta}$ is close to 1 for θ near 0, the function $\frac{\sin(\theta)}{\theta}\cdot\frac{1}{\theta}$ will behave like $\frac{1}{\theta}$, so the limit does not exist.

4. $\lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta}$

$$\lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}\cdot\sin(\theta) = 1\cdot 0 = 0$$

5. $\lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta^2}$

$\lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta^2} = 1$, since θ^2 is close to 0 when x is close to 0.

VII. (6) For the function $f(x) = x^3$:

1. Write $f(a+h)$ in the form $f(a)+mh+E(h)$ for some expression m involving only a and some function $E(h)$ of h. (Besides just rewriting the expression, tell explicitly what m equals, and what $E(h)$ equals in terms of h.)

$f(a+h) = (a+h)^3 = a^3+3a^2h+3ah^2+h^3 = f(a)+3a^2h+(3ah^2+h^3)$, so $m = 3a^2$ and $E(h) = 3ah^2+h^3$.

2. Find $\lim_{h\to 0} E(h)$, $\lim_{h\to 0} \frac{E(h)}{h}$, and $\lim_{h\to 0} \frac{E(h)}{h^2}$.

$$\lim_{h\to 0} 3ah^2+h^3 = 0$$
$$\lim_{h\to 0} \frac{3ah^2+h^3}{h} = \lim_{h\to 0} 3ah+h^2 = 0$$
$$\lim_{h\to 0} \frac{3ah^2+h^3}{h^2} = \lim_{h\to 0} 3a+h = 3a$$

VIII. (3) A certain function $f(x)$ satisfies $\lim_{x\to-\infty} f(x) = 5$. A second function $g(x)$ satisfies $\lim_{x\to5} g(x) = -\infty$. What is $\lim_{x\to-\infty} (g \circ f)(x)$, and why?

The limit is $-\infty$. The limit $\lim_{x\to5} g(x) = -\infty$ says that when z is a number near 5, $g(z)$ is a large negative number. When x is a large negative number, the value of $f(x)$ is near 5, because $\lim_{x\to-\infty} f(x) = 5$, so $g(f(x))$ is a large negative number. That is, $\lim_{x\to-\infty} (g \circ f)(x) = -\infty$.

IX. (5) Use the definition of limit to give a rigorous argument that $\lim_{x\to3} x^2 + x - 4 = 8$. Hint: Use the fact that $x^2 + x - 12 = (x+4)(x-3)$.

Let $\epsilon > 0$ be given. Put $\delta = \min\{1, \epsilon/8\}$. Suppose that x is any number such that $0 < |x-3| < \delta$. We have $|x+4| = |(x-3)+7| \le |x-3| + 7$, by the Triangle Inequality, so $|x+4| < \delta + 7 \le 8$. We then have

$$|(x^2+x-4) - 8| = |x^2+x-12| = |x-3| \cdot |x+4| \le |x-3| \cdot 8 < \delta \cdot 8 \le (\epsilon/8) \cdot 8 = \epsilon\ .$$

X. (2) Give a precise formal definition of $\lim_{x\to\infty} f(x) = -\infty$.

For every number N, there exists a number M such that if $x > M$ then $f(x) < N$.

XI. (2) State the Intermediate Value Theorem.

Suppose that f is a function continuous at every point in a closed interval $[a, b]$. If N is any number between $f(a)$ and $f(b)$, then there exists a number c between a and b such that $f(c) = N$.

XII. (2) For the function $x^2 + 1$ on the interval $[-5, 0]$ and the intermediate value $N = 11$, find all numbers whose existence is guaranteed by the Intermediate Value Theorem.

We want a number c so that $c^2 + 1 = 11$, which is satisfied when c is one of $\pm\sqrt{10}$. Only $-\sqrt{10}$ lies between -5 and 0, however, so the only number guaranteed by the IVT is $c = -\sqrt{10}$.

XIII. (3) Challenge Problem: Give an example of a function that is defined at every real number, but is continuous only at the point $x = 0$.

Define $f(x)$ to be x if x is rational, and $-x$ if x is irrational. We have $-x \le f(x) \le x$ for all x, so the Squeeze Theorem shows that $\lim_{x\to0} f(x) = 0 = f(0)$, and therefore f is continuous at 0. At any other a, some nearby values of f are close to a and some are close to $-a$, so $\lim_{x\to a} f(x)$ does not exist at any nonzero value of a.

II graphs: