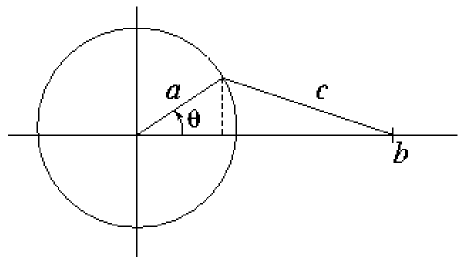
Mathematics 1823-001H Name (please print) ____________________

Final Examination

December 13, 2006

I. State the Law of Cosines, and verify it.
(5)

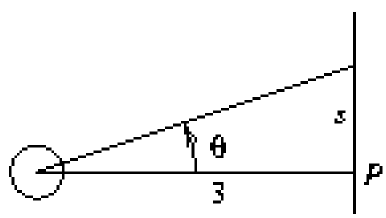
The Law of Cosines says that if a, b, and c are the sides of a triangle, and θ is the angle where the sides of lengths a and b meet, then $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.



To verify the Law of Cosines, we refer to the above figure. The coordinate of the vertex of the triangle that lies on the circle is $(a\cos(\theta), a\sin(\theta))$. Therefore the height of the triangle is $a\sin(\theta)$. The base of the right-hand right triangle is $b - a\cos(\theta)$. Applying the Pythagorean Theorem to that triangle, we have

$$c^2 = a^2\sin^2(\theta) + (b - a\cos(\theta))^2 = a^2\sin^2(\theta) + b^2 - 2ab\cos(\theta) + a^2\cos^2(\theta) = a^2 + b^2 - 2ab\cos(\theta)\ .$$

II. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute (that is, $\dfrac{d\theta}{dt} = 8\pi$ radians/min). How fast is the beam of light moving along the shoreline when it is 1 km from P?
(7)



We are given that $\dfrac{d\theta}{dt} = 8\pi$ and want to find $\dfrac{ds}{dt}$ when $s = 1$. Since $\tan(\theta) = s/3$, we have $\dfrac{ds}{dt} = 3\sec^2(\theta)\,\dfrac{d\theta}{dt}$. When $s = 1$, the hypotenuse of the triangle is $\sqrt{10}$ so $\sec(\theta) = \sqrt{10}/3$. Therefore at that moment, $\dfrac{ds}{dt} = 3(10/9)\,\dfrac{d\theta}{dt} = 80\pi/3$ km/min.

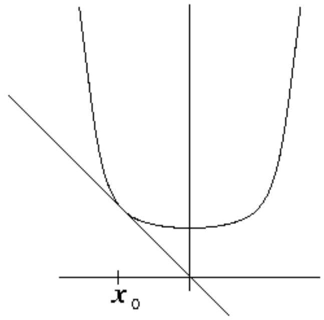
III. Find f if $f'''(t) = 60t^2$.
(4)

$f''(t) = 60t^3/3 + C_1 = 20t^3 + C_1$, $f'(t) = 20t^4/4 + C_1 t + C_2 = 5t^4 + C_1 t + C_2$, $f(t) = 5t^5/5 + C_1 t^2/2 + C_2 t + C_3 = t^5 + C_1 t^2/2 + C_2 t + C_3$

IV. Find f if $f''(x) = 2 + \cos(x)$, $f(0) = -1$, $f(\pi/2) = 0$.
(4)

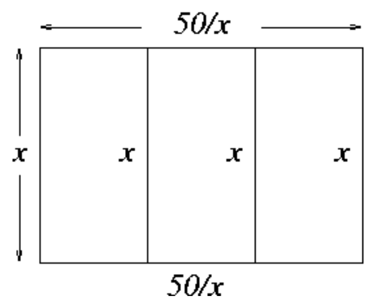
$f'(x) = 2x + \sin(x) + C_1$ and $f(x) = x^2 - \cos(x) + C_1 x + C_2$. When $x = 0$, this says $-1 = 0 - \cos(0) + C_2$, so $C_2 = 0$. When $x = \pi/2$, we then have $0 = \pi^2/4 - 0 + C_1\pi/2$, so $C_1 = -\pi/2$. Therefore $f(x) = x^2 - \cos(x) - \pi x/2$.

V. (5) Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f at some point.



We have $f(x) = x^4/4 + C$ for some constant C. Let $(x_0, x_0^4/4 + C)$ be the point of tangency. Then $x_0^3 = f'(x_0) = -1$, so $x_0 = -1$. Since the point of tangency is on $x + y = 0$, it must be $(-1, 1)$. Therefore $1 = (-1)^4/4 + C$, so $C = 3/4$ and $f(x) = x^4/4 + 3/4$.

VI. (7) A farmer wants to fence a rectangular area of 50 hectares and then divide it into thirds with two fences parellel to one of the sides of the field. What are the dimensions of the area that requires the least amount of fence? Remark: 1 hectare is $10{,}000$ square meters, so a length unit in this problem is 100 meters.



In the above diagram, the side has length x and the top has length $50/x$, since the total area is 50. The total length of fence is $F(x) = 4x + 50/x$, and we seek to minimize this subject to the condition that $x > 0$. We have $F'(x) = 4 - 100/x^2$, so there is one critical point $x = 5$. Since $F''(x) = 200/x^3$ is always positive on this domain, the graph is concave up and therefore $F(x)$ has an absolute mimimum when $x = 5$. So the solution is that x must be 5 and $50/x$ is 10. The length units are 100 meters, so the dimensions that use the least amount of fence are 500 meters by $1{,}000$ meters.

VII. (6) Recall that $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. For the function $f(x) = x^4$:

1. Write $f(a+h)$ in the form $f(a)+mh+E(h)$ for some expression m involving only a and some function $E(h)$ of h. (Besides just rewriting the expression, tell explicitly what m equals, and what $E(h)$ equals in terms of h.)

   $f(a + h) = (a + h)^4 = a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4 = f(a) + 4a^3h + (6a^2h^2 + 4ah^3 + h^4)$, so $m = 4a^3$ and $E(h) = 6a^2h^2 + 4ah^3 + h^4$.

2. Find $\lim_{h\to 0} E(h)$, $\lim_{h\to 0} \frac{E(h)}{h}$, and $\lim_{h\to 0} \frac{E(h)}{h^2}$.

$$\lim_{h\to 0} 6a^2h^2 + 4ah^3 + h^4 = 0$$
$$\lim_{h\to 0} \frac{6a^2h^2 + 4ah^3 + h^4}{h} = \lim_{h\to 0} 6ah + 4ah^2 + h^3 = 0$$
$$\lim_{h\to 0} \frac{6a^2h^2 + 4ah^3 + h^4}{h^2} = \lim_{h\to 0} 6a^2 + 4h + h^2 = 6a^2$$

VIII. (a) State the Mean Value Theorem.
(10)

The Mean Value Theorem states that if a function f is differentiable at all points between a and b, and is continuous at a and b as well, then there exists a c between a and b so that $f(b) - f(a) = f'(c)\,(b-a)$.

(b) Use the Mean Value Theorem to verify that a function with constant zero derivative on an interval must be constant.

Choose some point x_0 in the interval, and let x be any other point in the interval. Applying the Mean Value Theorem, we have $f(x) - f(x_0) = f'(c)(x - x_0)$. Since c is between x and x_0, it must also lie in the interval, and $f'(c) = 0$. So $f(x) = f(x_0)$. This is true for all x in the interval, so f is constant.

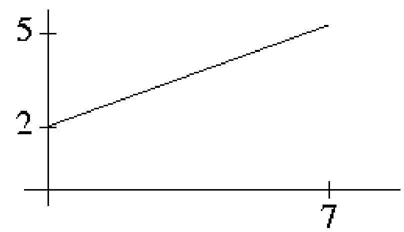
(c) Deduce that if two functions on an interval have the same derivative, then one of them is a constant plus the other one.

Writing f and g for the functions, we have $(f-g)' = f' - g' = 0$, so $f - g$ is constant, say $f(x) - g(x) = C$ for all x. So $f(x) = g(x) + C$ for all x.

(d) Show by example that (c) can be false if the domain of the functions is not a connected interval.

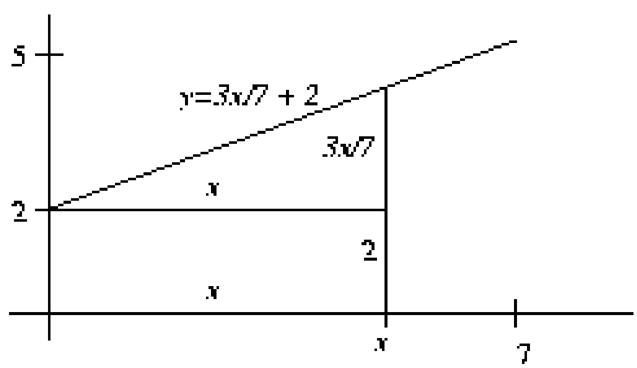
On the domain of nonzero real numbers, take for instance $f(x)$ to be 1 for $x > 0$ and 0 for $x < 0$, and $g(x)$ to be 0 for every nonzero x. Both f and g have derivative the 0 function, but for $x > 0$ they differ by 1 and for $x < 0$ they differ by 0.

IX. To the right is the graph of a linear function f with domain the interval $[0, 7]$ and range the interval $[2, 5]$.
(5)



(a) Find an explicit expression for $A(x)$, the area under the graph of f between 0 and x.

(b) Verify by calculation that $A'(x) = f(x)$.



In the above diagram, the function is $f(x) = 3x/7 + 2$, and area beneath its graph between 0 and x breaks into a rectangle of area $2x$ and a triangle of area $(1/2) \cdot x \cdot (3x/7) = 3x^2/14$. So the area function is $A(x) = 3x^2/14 + 2x$, and $A'(x) = 3(2x)/14 + 2 = 3x/7 + 2 = f(x)$.

X. State the Intermediate Value Theorem.
(3)

The Intermediate Value Theorems states that if $f(x)$ is a continuous function on a closed interval $[a, b]$, and r is any value between a and b, then there exists a value c between a and b for which $f(c) = r$.

XI. (7) Using calculus, graph the function $2x - \tan(x)$ for $-\pi/2 < x < \pi/2$. Indicate critical points and inflection points, if any, and any asymptotic behavior.

Write $f(x)$ for this function.

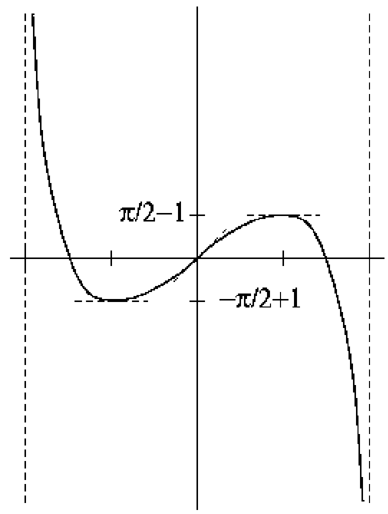
Both $2x$ and $\tan(x)$ are odd functions, so $f(x)$ is odd.

We have $f'(x) = 2 - \sec^2(x)$ and $f''(x) = -2\sec^2(x)\tan(x)$. The critical points are where $\sec^2(x) = 2$, $\sec(x) = \pm\sqrt{2}$, $\cos(x) = \pm 1/\sqrt{2}$, $x = \pm\pi/4$. At the critical points, the values of f are $f(\pi/4) = \pi/2 - 1$ and $f(-\pi/4) = -\pi/2 + 1$.

For inflection points, since $\sec^2(x) > 0$, we find that $f''(x) > 0$ for $-\pi/2 < x < 0$ and $f''(x) < 0$ for $0 < x < \pi/2$, so there is an inflection point at $x = 0$ where $f(0) = 0$. And the slope at the origin if $f'(0) = 2 - 1 = 1$.

As x approaches $\pi/2$, the value of $f(x)$ goes to $-\infty$.

We now have a good idea of what the graph looks like:



XII. (9) Determine the following limits (including the possible values ∞ and $-\infty$) by using algebraic manipulation to put them into a form where the limit is obvious.

(a) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+2}}$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+2}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{1+2/x^2}} = \lim_{x\to\infty} \frac{x}{x\sqrt{1+2/x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+2/x^2}} = \frac{1}{\sqrt{1+0}} = 1$$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+2}}$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+2}} = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2}\sqrt{1+2/x^2}} = \lim_{x\to-\infty} \frac{x}{-x\sqrt{1+2/x^2}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+2/x^2}}$$
$$= \frac{1}{-\sqrt{1+0}} = -1$$

(c) $\displaystyle\lim_{x\to-\infty} x^{12} - x^{13}$

$\displaystyle\lim_{x\to-\infty} x^{12} - x^{13} = \lim_{x\to-\infty} x^{13}(1/x - 1)$. Since $\displaystyle\lim_{x\to-\infty} 1/x - 1 = -1$, the value of $x^{13}(1/x - 1))$ for large negative x is close to the value of $-x^{13}$, so $\displaystyle\lim_{x\to-\infty} x^{12} - x^{13} = \infty$.

XIII. Give a precise formal definition of $\lim_{\delta \to L} G(\delta) = x$.
(2)

For every $\epsilon > 0$, there exists $\omega > 0$ so that if $0 < |\delta - L| < \omega$, then $|G(\delta) - x| < \epsilon$.

XIV. Give a precise formal definition of $\lim_{\delta \to -\infty} G(\delta) = \infty$.
(2)

For every M, there exists N so that if $\delta < N$, then $G(\delta) > M$.

XV. Challenge Problem: Give an explicit example of a continuous function on the interval $(0, 1]$ that has no minimum value and no maximum value.
(3)

Let $f(x) = \sin(1/x)/x$. Since $\sin(1/x)$ takes on the value 1 infinitely many times as x approaches 0, $f(x)$ equals $1/x$ for infinitely many x-values close to 0, so $f(x)$ has no maximum value. Similarly, $\sin(1/x)$ takes on the value -1 infinitely many times as x approaches 0, and $f(x)$ equals $-1/x$ at all such x-values, so $f(x)$ has no mimimum value.