Mathematics 2513-002 Name (please print) ____________________

Examination III

November 14, 2005

Instructions: Give *brief,* clear answers.

I. Prove that the function $f\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ defined by $f(m,n)=m-n$ is surjective.
(4)

II. Using the notation $h\colon Y\to X$, define the *range* of h, the *preimage* of x for an element $x\in X$, the *image*
(4) of y for an element $y\in Y$, and the *graph* of h.

III. Let S be the set of sequences of 0's and 1's, $S=\{a_1a_2a_3\cdots \mid a_i\in\{0,1\}\}$. A typical element of S is
(5) $001011011100011010\cdots$. Adapt Cantor's proof that $\mathbb{R}$ is uncountable to prove that S is uncountable.

IV. Let a and b be integers, at least one of them nonzero.
(11)

1. Define the *greatest common divisor* $\gcd(a,b)$.
2. Find $\gcd(2^3\cdot 3^3\cdot 7^2\cdot 13\cdot 17, 2\cdot 3^4\cdot 5\cdot 7\cdot 17)$ and $\mathrm{lcm}(2^3\cdot 3^3\cdot 7^2\cdot 13\cdot 17, 2\cdot 3^4\cdot 5\cdot 7\cdot 17)$ (leave the results in factored form, do not multiply them out).
3. Describe the *Euclidean algorithm* for computing $\gcd(a,b)$.

V. Which positive integers less than 10 are relatively prime to 10?
(3)

VI. Use the fact that $7\cdot 8\equiv 1 \mod 55$ to find an integer m for which $8m\equiv 11 \mod 55$.
(4)

VII. Prove that $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ whenever n is a positive integer.
(5)

VIII. Prove that if there exists d so that $cd\equiv 1 \mod m$, then $\gcd(c,m)=1$. Hint: use the theorem that says
(5) $\gcd(a,b)$ is the least positive sum of multiples of a and b.

IX. Adapt the argument of Cantor's proof that $\mathbb{Q}$ is countable to prove that $\mathbb{N}\times\mathbb{N}$ is countable.
(5)

X. Use congruence to prove that 3 divides n^3+2n for any integer n.
(5)