I. Prove that the function \( f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) defined by \( f(m, n) = m - n \) is surjective. (4)

II. Using the notation \( h: Y \to X \), define the range of \( h \), the preimage of \( x \) for an element \( x \in X \), the image of \( y \) for an element \( y \in Y \), and the graph of \( h \). (4)

III. Let \( S \) be the set of sequences of 0's and 1's, \( S = \{a_1a_2a_3\cdots \mid a_i \in \{0, 1\}\} \). A typical element of \( S \) is 001011011100011010\cdots. Adapt Cantor’s proof that \( \mathbb{R} \) is uncountable to prove that \( S \) is uncountable. (5)

IV. Let \( a \) and \( b \) be integers, at least one of them nonzero. (11)
   1. Define the greatest common divisor \( \gcd(a, b) \).
   2. Find \( \gcd(2^3 \cdot 3^3 \cdot 7^2 \cdot 13 \cdot 17, 2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17) \) and \( \text{lcm}(2^3 \cdot 3^3 \cdot 7^2 \cdot 13 \cdot 17, 2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 17) \) (leave the results in factored form, do not multiply them out).
   3. Describe the Euclidean algorithm for computing \( \gcd(a, b) \).

V. Which positive integers less than 10 are relatively prime to 10? (3)

VI. Use the fact that \( 7 \cdot 8 \equiv 1 \mod 55 \) to find an integer \( m \) for which \( 8m \equiv 11 \mod 55 \). (4)

VII. Prove that \( 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1 \) whenever \( n \) is a positive integer. (5)

VIII. Prove that if there exists \( d \) so that \( cd \equiv 1 \mod m \), then \( \gcd(c, m) = 1 \). Hint: use the theorem that says \( \gcd(a, b) \) is the least positive sum of multiples of \( a \) and \( b \). (5)

IX. Adapt the argument of Cantor’s proof that \( \mathbb{Q} \) is countable to prove that \( \mathbb{N} \times \mathbb{N} \) is countable. (5)

X. Use congruence to prove that 3 divides \( n^3 + 2n \) for any integer \( n \). (5)