Instructions: Give brief, clear answers.

I. For sets $A$ and $B$, give the precise definitions of $A \cap B$, $A \cup B$, $A \subseteq B$, and $A = B$.
1. $A \cap B = \{x \mid x \in A \land x \in B\}$.
2. $A \cup B = \{x \mid x \in A \lor x \in B\}$.
3. $A \subseteq B \equiv \forall x, (x \in A \Rightarrow x \in B)$.
4. $A = B \equiv \forall x, (x \in A \leftrightarrow x \in B)$.

II. Prove that $\{\emptyset, \{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ is false.
3. $\{\emptyset\} \in \emptyset, \{\emptyset\}\}$ but $\{\emptyset\} \notin \emptyset, \{\emptyset\}$.

III. Disprove the following assertions.
1. For any three sets $A$, $B$, and $C$, if $A \cup C = B \cup C$, then $A = B$.
   
   $\mathbb{N} \cup \mathbb{R} = \mathbb{R} \cup \mathbb{R}$ but $\mathbb{N} \neq \mathbb{R}$, or
   
   $\{1\} \cup \{1, 2\} = \{2\} \cup \{1, 2\}$, but $\{1\} \neq \{2\}$.

2. For any three sets $A$, $B$, and $C$, $A \cup (B \cap C) = (A \cup B) \cap C$.
   
   $\mathbb{R} \cup (\mathbb{N} \cap \mathbb{Z}) = \mathbb{R} \cup \mathbb{N} = \mathbb{R}$ but $(\mathbb{R} \cup \mathbb{N}) \cap \mathbb{Z} = \mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$, or
   
   $\{1\} \cup (\{1\} \cap \{2\}) = \{1\} \cup \emptyset = \{1\}$, but $(\{1\} \cup \{1\}) \cap \{2\} = \{1\} \cap \{2\} = \emptyset$.

IV. Prove that if $A \subseteq B$, then $A \times C \subseteq B \times C$.
4. Assume that $A \subseteq B$. Assume that $(a, c) \in A \times C$, so $a \in A$ and $c \in C$. Since $A \subseteq B$, we have $a \in B$. Since $a \in B$ and $c \in C$, $(a, c) \in B \times C$.

V. Prove that the function $f \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(m, n) = m - n$ is surjective.
3. Let $k \in \mathbb{Z}$. Then, $(k, 0) \in \mathbb{Z} \times \mathbb{Z}$ and $f(k, 0) = k$.

VI. Prove that the function $g \colon [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^2$ is injective.
4. Let $r_1, r_2 \in [0, \infty)$ and assume that $r_1^2 = r_2^2$. Then $\sqrt{r_1^2} = \sqrt{r_2^2}$, that is, $|r_1| = |r_2|$. Since $r_1 \geq 0$, we have $|r_1| = r_1$, and similarly $|r_2| = r_2$, so $r_1 = r_2$.

VII. State Rolle’s Theorem. Use it to give a proof by contradiction showing that the function $f \colon [0, \pi] \to [-1, 1]$ defined by $f(x) = \cos(x)$ is injective.
5. Rolle’s Theorem says that if a function $f \colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$, and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Suppose for contradiction that there exist $x_1, x_2 \in [0, \pi]$ with $\cos(x_1) = \cos(x_2)$ but $x_1 \neq x_2$. By Rolle’s Theorem, there exists $c$ between $x_1$ and $x_2$ for which $0 = \cos'(c) = -\sin(c)$. But $\sin(c) \neq 0$ for any $c \in (0, \pi)$, a contradiction.
VIII. For the function \( f: \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \pi x - 13.4 \), find a formula for the composition \((f \circ f \circ f)(x)\).

\[
(f \circ f \circ f)(x) = f(f(f(x))) = f(f(\pi x - 13.4)) = f(\pi(\pi x - 13.4) - 13.4) = f(\pi^2 x - 13.4\pi - 13.4) = \pi(\pi^2 x - 13.4\pi - 13.4) - 13.4 = \pi^3 x - 13.4\pi^2 - 13.4\pi - 13.4.
\]

IX. Using the notation \( h: Y \to X \), define the range of \( h \), the preimage of \( x \) for an element \( x \in X \), the image of \( y \) for an element \( y \in Y \), and the graph of \( h \).

The range of \( h \) is \( \{ x \in X \mid \exists y \in Y, h(y) = x \} \), or \( \{ h(y) \mid y \in Y \} \).

The preimage of \( x \) is \( \{ y \in Y \mid h(y) = x \} \).

The image of \( y \) is \( h(y) \).

The graph of \( h \) is the set \( \{ (y, h(y)) \mid y \in Y \} \) (or \( \{ (y, x) \in Y \times X \mid x = h(y) \} \)).

X. Simplify each of the following:

1. \( (\overline{2, \infty}) \cap (0, 3] \), assuming that the universal set is \( \mathcal{U} = \mathbb{R} \) (the answer should be written as a union of two intervals).

\[
\overline{2, \infty}) \cap (0, 3] = (-\infty, 2] \cap (0, 3] = (0, 2] = (-\infty, 0] \cup (2, \infty), \text{ or}
\]
\[
\overline{2, \infty}) \cap (0, 3] = (2, \infty) \cup ((-\infty, 0] \cup (3, \infty)) = (-\infty, 0] \cup ((2, \infty) \cup (3, \infty))
\]

2. \( P(0) \cap P(1) \), where \( P(r) \) denotes the preimage of a number \( r \) for a function \( f: \mathbb{R} \to \mathbb{R} \).

\[
P(0) \cap P(1) = \{ x \in \mathbb{R} \mid f(r) = 0 \} \cap \{ x \in \mathbb{R} \mid f(r) = 1 \} = \{ x \in \mathbb{R} \mid f(r) = 0 \land f(r) = 1 \} = \emptyset
\]

XI. Prove that if \( a|b \) and \( b|c \), then \( a|c \).

Assume that \( a|b \) and \( b|c \). Then there exist integers \( k, \ell \) so that \( b = ka \) and \( c = \ell b \). So \( c = \ell b = (\ell k)a \), that is, \( a|c \).

XII. Prove that if \( a|c \) and \( b|d \), then \( ab|cd \).

Assume that \( a|c \) and \( b|d \). Then there exist integers \( k, \ell \) so that \( c = ka \) and \( d = \ell b \). So we have \( cd = (ka)(\ell b) = (k\ell)ab \), that is, \( ab|cd \).

XIII. State the Fundamental Theorem of Arithmetic.

Any integer \( a > 1 \) can be written as a product of prime factors. If the factors are written in nondecreasing order, then this factorization is unique.

XIV. Complete the following proof that there are infinitely many primes: “Suppose for contradiction that there are finitely many primes, say \( p_1, p_2, \ldots, p_k \). Put \( N = p_1 p_2 \cdots p_k + 1 \). Notice that no \( p_i \) divides \( N \). . .”

If \( N \) is prime, then it is a prime different from any of the \( p_i \), a contradiction. If \( N \) is composite, write it as \( N = q_1 q_2 \cdots q_m \). Then \( q_1 \) is a prime which divides \( N \), so \( q_1 \) is a prime which is not equal to any of the \( p_i \), again a contradiction.