Mathematics 2513-002 Name (please print) ____________________

Examination II

October 17, 2005

Instructions: Give *brief,* clear answers.

I. (4) For sets A and B, give the precise definitions of $A \cap B$, $A \cup B$, $A \subseteq B$, and $A = B$.

1. $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
2. $A \cup B = \{x \mid x \in A \vee x \in B\}$.
3. $A \subseteq B \equiv \forall x, (x \in A \Rightarrow x \in B)$.
4. $A = B \equiv \forall x, (x \in A \Leftrightarrow x \in B)$.

II. (3) Prove that $\{\emptyset, \{\{\emptyset\}\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ is false.

$\{\{\emptyset\}\} \in \{\emptyset, \{\{\emptyset\}\}\}$ but $\{\{\emptyset\}\} \notin \{\emptyset, \{\emptyset\}\}$.

III. (4) Disprove the following assertions.

1. For any three sets A, B, and C, if $A \cup C = B \cup C$, then $A = B$.

   $\mathbb{N} \cup \mathbb{R} = \mathbb{R} = \mathbb{Z} \cup \mathbb{R}$ but $\mathbb{N} \neq \mathbb{Z}$, or
   $\{1\} \cup \{1, 2\} = \{2\} \cup \{1, 2\}$, but $\{1\} \neq \{2\}$.

2. For any three sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap C$.

   $\mathbb{R} \cup (\mathbb{N} \cap \mathbb{Z}) = \mathbb{R} \cup \mathbb{N} = \mathbb{R}$ but $(\mathbb{R} \cup \mathbb{N}) \cap \mathbb{Z} = \mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$, or
   $\{1\} \cup (\{1\} \cap \{2\}) = \{1\} \cup \emptyset = \{1\}$, but $(\{1\} \cup \{1\}) \cap \{2\} = \{1\} \cap \{2\} = \emptyset$.

IV. (4) Prove that if $A \subseteq B$, then $A \times C \subseteq B \times C$.

Assume that $A \subseteq B$. Assume that $(a, c) \in A \times C$, so $a \in A$ and $c \in C$. Since $A \subseteq B$, we have $a \in B$. Since $a \in B$ and $c \in C$, $(a, c) \in B \times C$.

V. (3) Prove that the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(m, n) = m - n$ is surjective.

Let $k \in \mathbb{Z}$. Then, $(k, 0) \in \mathbb{Z} \times \mathbb{Z}$ and $f(k, 0) = k$.

VI. (4) Prove that the function $g\colon [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^2$ is injective.

Let $r_1, r_2 \in [0, \infty)$ and assume that $r_1^2 = r_2^2$. Then $\sqrt{r_1^2} = \sqrt{r_2^2}$, that is, $|r_1| = |r_2|$. Since $r_1 \geq 0$, we have $|r_1| = r_1$, and similarly $|r_2| = r_2$, so $r_1 = r_2$.

VII. (5) State Rolle's Theorem. Use it to give a proof by contradiction showing that the function $f\colon [0, \pi] \to [-1, 1]$ defined by $f(x) = \cos(x)$ is injective.

Rolle's Theorem says that if a function $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Suppose for contradiction that there exist $x_1, x_2 \in [0, \pi]$ with $\cos(x_1) = \cos(x_2)$ but $x_1 \neq x_2$. By Rolle's Theorem, there exists c between x_1 and x_2 for which $0 = \cos'(c) = -\sin(c)$. But $\sin(c) \neq 0$ for any $c \in (0, \pi)$, a contradiction.

VIII. (3) For the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \pi\, x - 13.4$, find a formula for the composition $(f \circ f \circ f)(x)$.

$(f \circ f \circ f)(x) = f(f(f(x))) = f(f(\pi x - 13.4)) = f(\pi(\pi x - 13.4) - 13.4) = f(\pi^2 x - 13.4\pi - 13.4) = \pi(\pi^2 x - 13.4\pi - 13.4) - 13.4 = \pi^3 x - 13.4\pi^2 - 13.4\pi - 13.4$.

IX. (4) Using the notation $h\colon Y \to X$, define the *range* of h, the *preimage* of x for an element $x \in X$, the *image* of y for an element $y \in Y$, and the *graph* of h.

The range of h is $\{x \in X \mid \exists y \in Y, h(y) = x\}$, or $\{h(y) \mid y \in Y\}$.

The preimage of x is $\{y \in Y \mid h(y) = x\}$.

The image of y is $h(y)$.

The graph of h is the set $\{(y, h(y)) \mid y \in Y\}$ (or $\{(y, x) \in Y \times X \mid x = h(y)\}$).

X. (4) Simplify each of the following:

1. $\overline{\overline{(2,\infty)} \cap (0,3]}$, assuming that the universal set is $\mathcal{U} = \mathbb{R}$ (the answer should be written as a union of two intervals).

   $\overline{\overline{(2,\infty)} \cap (0,3]} = \overline{(-\infty,2] \cap (0,3]} = \overline{(0,2]} = (-\infty,0] \cup (2,\infty)$, or
   $\overline{\overline{(2,\infty)} \cap (0,3]} = \overline{\overline{(2,\infty)}} \cup \overline{(0,3]} = (2,\infty) \cup ((-\infty,0] \cup (3,\infty)) = (-\infty,0] \cup ((2,\infty) \cup (3,\infty))$
   $= (-\infty,0] \cup (2,\infty)$

2. $P(0) \cap P(1)$, where $P(r)$ denotes the preimage of a number r for a function $f\colon \mathbb{R} \to \mathbb{R}$.

   $P(0) \cap P(1) = \{x \in \mathbb{R} \mid f(r) = 0\} \cap \{x \in \mathbb{R} \mid f(r) = 1\} = \{x \in \mathbb{R} \mid f(r) = 0 \wedge f(r) = 1\} = \emptyset$

XI. (4) Prove that if $a|b$ and $b|c$, then $a|c$.

Assume that $a|b$ and $b|c$. Then there exist integers k, ℓ so that $b = ka$ and $c = \ell b$. So $c = \ell b = (\ell k)a$, that is, $a|c$.

XII. (4) Prove that if $a|c$ and $b|d$, then $ab|cd$.

Assume that $a|c$ and $b|d$. Then there exist integers k, ℓ so that $c = ka$ and $d = \ell b$. So we have $cd = (ka)(\ell b) = (k\ell)ab$, that is, $ab|cd$.

XIII. (3) State the Fundamental Theorem of Arithmetic.

Any integer $a > 1$ can be written as a product of prime factors. If the factors are written in nondecreasing order, then this factorization is unique.

XIV. (4) Complete the following proof that there are infinitely many primes: "Suppose for contradiction that there are finitely many primes, say $p_1, p_2, \ldots, p_k$. Put $N = p_1 p_2 \cdots p_k + 1$. Notice that no p_i divides N. ..."

If N is prime, then it is a prime different from any of the p_i, a contradiction. If N is composite, write it as $N = q_1 q_2 \cdots q_m$. Then q_1 is a prime which divides N, so q_1 is a prime which is not equal to any of the p_i, again a contradiction.