Mathematics 2513-002 Name (please print) ____________________

Examination II

October 17, 2005

Instructions: Give *brief,* clear answers.

I. (4) For sets A and B, give the precise definitions of $A \cap B$, $A \cup B$, $A \subseteq B$, and $A = B$.

II. (3) Prove that $\{\emptyset, \{\{\emptyset\}\}\} \subseteq \{\emptyset, \{\emptyset\}\}$ is false.

III. (4) Disprove the following assertions.

1. For any three sets A, B, and C, if $A \cup C = B \cup C$, then $A = B$.
2. For any three sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap C$.

IV. (4) Prove that if $A \subseteq B$, then $A \times C \subseteq B \times C$.

V. (3) Prove that the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(m, n) = m - n$ is surjective.

VI. (4) Prove that the function $g\colon [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^2$ is injective.

VII. (5) State Rolle's Theorem. Use it to give a proof by contradiction showing that the function $f\colon [0, \pi] \to [-1, 1]$ defined by $f(x) = \cos(x)$ is injective.

VIII. (3) For the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \pi\, x - 13.4$, find a formula for the composition $(f \circ f \circ f)(x)$.

IX. (4) Using the notation $h\colon Y \to X$, define the *range* of h, the *preimage* of x for an element $x \in X$, the *image* of y for an element $y \in Y$, and the *graph* of h.

X. (4) Simplify each of the following:

1. $\overline{\overline{(2, \infty)} \cap (0, 3]}$, assuming that the universal set is $\mathcal{U} = \mathbb{R}$ (the answer should be written as a union of two intervals).
2. $P(0) \cap P(1)$, where $P(r)$ denotes the preimage of a number r for a function $f\colon \mathbb{R} \to \mathbb{R}$.

XI. (4) Prove that if $a|b$ and $b|c$, then $a|c$.

XII. (4) Prove that if $a|c$ and $b|d$, then $ab|cd$.

XIII. (3) State the Fundamental Theorem of Arithmetic.

XIV. (4) Complete the following proof that there are infinitely many primes: "Suppose for contradiction that there are finitely many primes, say $p_1, p_2, \ldots, p_k$. Put $N = p_1 p_2 \cdots p_k + 1$. Notice that no p_i divides N. ..."