Instructions: Give brief, clear answers.

I. For sets \( A \) and \( B \), give the precise definitions of \( A \cap B \), \( A \cup B \), \( A \subseteq B \), and \( A = B \).

II. Prove that \( \emptyset, \{ \emptyset \} \subseteq \{ \emptyset, \{ \emptyset \} \} \) is false.

III. Disprove the following assertions.

1. For any three sets \( A, B, \) and \( C \), if \( A \cup C = B \cup C \), then \( A = B \).

2. For any three sets \( A, B, \) and \( C \), \( A \cup (B \cap C) = (A \cup B) \cap C \).

IV. Prove that if \( A \subseteq B \), then \( A \times C \subseteq B \times C \).

V. Prove that the function \( f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) defined by \( f(m, n) = m - n \) is surjective.

VI. Prove that the function \( g: [0, \infty) \to \mathbb{R} \) defined by \( g(x) = x^2 \) is injective.

VII. State Rolle’s Theorem. Use it to give a proof by contradiction showing that the function \( f: [0, \pi] \to [-1, 1] \) defined by \( f(x) = \cos(x) \) is injective.

VIII. For the function \( f: \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \pi x - 13.4 \), find a formula for the composition \( f \circ f \circ f(x) \).

IX. Using the notation \( h: Y \to X \), define the range of \( h \), the preimage of \( x \) for an element \( x \in X \), the image of \( y \) for an element \( y \in Y \), and the graph of \( h \).

X. Simplify each of the following:

1. \( (2, \infty) \cap (0, 3] \), assuming that the universal set is \( \mathcal{U} = \mathbb{R} \) (the answer should be written as a union of two intervals).

2. \( P(0) \cap P(1) \), where \( P(r) \) denotes the preimage of a number \( r \) for a function \( f: \mathbb{R} \to \mathbb{R} \).

XI. Prove that if \( a | b \) and \( b | c \), then \( a | c \).

XII. Prove that if \( a | c \) and \( b | d \), then \( ab | cd \).

XIII. State the Fundamental Theorem of Arithmetic.

XIV. Complete the following proof that there are infinitely many primes: “Suppose for contradiction that there are finitely many primes, say \( p_1, p_2, \ldots, p_k \). Put \( N = p_1 p_2 \cdots p_k + 1 \). Notice that no \( p_i \) divides \( N \). . . ”