Mathematics 2513-002 Name (please print) ____________________

Examination I

September 16, 2005

Instructions: Give brief, clear answers.

I. Use a truth table to verify the tautology $((P \vee Q) \wedge \neg Q) \Rightarrow P$.
(6)

P	Q	$P \vee Q$	$\neg Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

II. Use the formal negation of quantified statements to simplify the following statements. In part (b), use
(6) the set of people in the class as the universal set, write the statement symbolically as the negation of an existentially quantified statement, simplify it to a universally quantified statement, and express the result in everyday language.

(a) $\neg\forall x \in \mathbb{R}, (x \leq 5 \Rightarrow f(x) > 2)$

$$\begin{aligned}
&\neg\forall x \in \mathbb{R}, (x \leq 5 \Rightarrow f(x) > 2)\\
&\equiv \exists x \in \mathbb{R}, \neg(x \leq 5 \Rightarrow f(x) > 2)\\
&\equiv \exists x \in \mathbb{R}, \neg(x > 5 \vee f(x) > 2)\\
&\equiv \exists x \in \mathbb{R}, (x \leq 5 \wedge f(x) \leq 2)
\end{aligned}$$

(b) There is no one in the class who knows both French and Russian.

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$\equiv \neg\exists x, (x \text{ knows French } \wedge x \text{ knows Russian })$

$\equiv \forall x, \neg(x \text{ knows French } \wedge x \text{ knows Russian })$

$\equiv \forall x, (\neg(x \text{ knows French }) \vee \neg(x \text{ knows Russian }))$

$\equiv$ Each person in the class either does not know French or does not know Russian

III. Express each of the following statements in mathematical notation using quantifiers:
(6)

(a) Every positive integer is the sum of the squares of four integers.

$$\forall n \in \mathbb{N}, \exists p \in \mathbb{Z}, \exists q \in \mathbb{Z}, \exists r \in \mathbb{Z}, \exists s \in \mathbb{Z}, n = p^2 + q^2 + r^2 + s^2$$

(b) There is a positive integer that is not the sum of the squares of three integers. (Simplify any negated quantifiers.)

$$\exists n \in \mathbb{N}, \forall p \in \mathbb{Z}, \forall q \in \mathbb{Z}, \forall r \in \mathbb{Z}, n \neq p^2 + q^2 + r^2$$

IV. (8) For each of the following statements, determine whether the statement is true or false, and briefly explain why. In both statements, the universal set for both x and y is $\mathbb{R}$.

(a) $\forall x, (x \neq 0 \Rightarrow (\exists y, xy = 1))$

It is true, since for any nonzero x we can choose y to be $1/x$.

(b) $\forall x, \exists y, ((x + y = 2) \wedge (2x - y = 1))$

It is false, since for $x = 0$, y would have to satisfy both $y = 2$ and $y = -1$.

V. (6) Write the following assertion as a universally quantified statement, then prove that it is true: For any integer n, if n is greater than 1, then $n^2 > n$.

$\forall n \in \mathbb{Z}, n > 1 \Rightarrow n^2 > n$

Let n be any integer. Assume that $n > 1$. Since $n > 0$, multiplying both sides by n preserves the inequality, so $n \cdot n > n \cdot 1$, that is, $n^2 > n$.

VI. (6) Write the following assertion as a universally quantified statement (use $\mathbb{R} - \mathbb{Q}$ to denote the set of irrational real numbers), then prove that it is false: The product of irrational real numbers is irrational.

$\forall x \in \mathbb{R} - \mathbb{Q}, \forall y \in \mathbb{R} - \mathbb{Q}, xy \in \mathbb{R} - \mathbb{Q}$

We will give a counterexample. Take $x = \sqrt{2}$ and $y = 1/\sqrt{2}$. By a result of Pythagoras, $\sqrt{2}$ is irrational, and $1/\sqrt{2}$ is irrational since if it were rational, say $1/\sqrt{2} = p/q$, then we would have $\sqrt{2} = q/p$, contradicting the fact that $\sqrt{2}$ is irrational. But $\sqrt{2} \cdot (1/\sqrt{2}) = 1$, which is rational.

VII. (6) The general form of a proof of an implication $P \Rightarrow Q$ by direct argument is this:

Statement: $P \Rightarrow Q$.
proof: Assume P.
...
Therefore Q. □

Give a similar outline of the structure of a proof by contradiction. That is, suppose that the statement to be proven is P. The argument begins by assuming that $\neg P$ is true (i. e. that P is false). Then what happens? Explain why this reasoning shows that P is true.

Statement: P.
proof: Assume $\neg P$.
...
Therefore Q.
But Q is false [or, $\neg Q$ is true]
Therefore P. □

The first section of the argument proves the implication $\neg P \Rightarrow Q$, by a direct argument. This is equivalent to its contrapositive, $\neg Q \Rightarrow P$. Then, one observes that $\neg Q$ is true, so the implication $\neg Q \Rightarrow P$ guarantees that P is true.

VIII. (6) Use proof by contradiction to prove that the sum of a rational number and an irrational number is irrational.

Suppose for contradiction that there exist a rational number x and an irrational number y so that $x + y$ is rational. We can write $x = p/q$ and $x + y = r/s$ for some integers p, q, r, and s. But then, $y = (x+y) - x = r/s - p/q = (rq - ps)/sq$, so y is rational, contradicting the fact that y was irrational. Therefore x and y cannot exist.