Mathematics 2513-002 Name (please print) ________________

Examination I

September 16, 2005

Instructions: Give brief, clear answers.

I. (6) Use a truth table to verify the tautology $((P \vee Q) \wedge \neg Q) \Rightarrow P$.

II. (6) Use the formal negation of quantified statements to simplify the following statements. In part (b), use the set of people in the class as the universal set, write the statement symbolically as the negation of an existentially quantified statement, simplify it to a universally quantified statement, and express the result in everyday language.

(a) $\neg\forall x \in \mathbb{R}, (x \leq 5 \Rightarrow f(x) > 2)$

(b) There is no one in the class who knows both French and Russian.

III. (6) Express each of the following statements in mathematical notation using quantifiers:

(a) Every positive integer is the sum of the squares of four integers.

(b) There is a positive integer that is not the sum of the squares of three integers. (Simplify any negated quantifiers.)

IV. (8) For each of the following statements, determine whether the statement is true or false, and briefly explain why. In both statements, the universal set for both x and y is $\mathbb{R}$.

(a) $\forall x, (x \neq 0 \Rightarrow (\exists y, xy = 1))$

(b) $\forall x, \exists y, ((x + y = 2) \wedge (2x - y = 1))$

V. (6) Write the following assertion as a universally quantified statement, then prove that it is true: For any integer n, if n is greater than 1, then $n^2 > n$.

VI. (6) Write the following assertion as a universally quantified statement (use $\mathbb{R} - \mathbb{Q}$ to denote the set of irrational real numbers), then prove that it is false: The product of irrational real numbers is irrational.

VII. (6) The general form of a proof of an implication $P \Rightarrow Q$ by direct argument is this:

Statement: $P \Rightarrow Q$.
proof: Assume P.
...
Therefore Q. $\Box$

Give a similar outline of the structure of a proof by contradiction. That is, suppose that the statement to be proven is P. The argument begins by assuming that $\neg P$ is true (i. e. that P is false). Then what happens? Explain why this reasoning shows that P is true.

VIII. (6) Use proof by contradiction to prove that the sum of a rational number and an irrational number is irrational.