Mathematics 2513-002 Name (please print) ____________________

Final Examination

December 14, 2005

I. (4) Use a direct argument to prove the following implication: "The product of two odd integers is odd."

II. (4) Prove the following implication by proving its contrapositive: "If $n^3 \not\equiv 2 \mod 3$, then $n \not\equiv 2 \mod 3$" (where $\not\equiv$ means "is not congruent to").

III. (4) Fill in the missing parts of the following proof by contradiction: [fill in] If N is prime, then it is a prime different from any of the p_i, a contradiction. If N is composite, write it as $N = q_1 q_2 \cdots q_m$. [fill in]

IV. (3) Disprove the following assertion: "For all integers a, b, and n, if $n|ab$, then $n|a$ or $n|b$."

V. (4) Using the notation $G\colon X_2 \to X_1$, define the *range* of G, the *preimage* of x_1 for an element $x_1 \in X_1$, the *image* of x_2 for an element $x_2 \in X_2$, and the *graph* of G.

VI. (5) Prove that the set S of all two-element subsets of $\mathbb{N}$ is countable as follows: List all the two-element subsets that contain 1 as follows: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\} \ldots$. In a second row list all the subsets that contain 2 but do not contain 1 as: $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{2,6\} \ldots$. Continue listing a third row and fourth row, so that if the process of writing rows were continued, every two-element subset would appear exactly once. Now describe how to form a single list in which every two-element subset appears exactly once.

VII. (5) Let S be the set of functions from $\mathbb{N}$ to $\mathbb{N}$, that is, $S = \{f \mid f\colon \mathbb{N} \to \mathbb{N}\}$. Fill in the missing items in the following proof that S is uncountable: "Suppose for [fill in] that S is countable. Then, the elements of $\mathbb{N}$ can be listed as f_1, $f_2, \ldots$. Define a new function $g \in S$, that is, $g\colon \mathbb{N} \to \mathbb{N}$, as follows: For each $n \in \mathbb{N}$, let [fill in] if $f_n(n) = 1$ and let [fill in] if $f_n(n) \neq 1$. Then for each n, $g(n) \neq f_n(n)$ so [fill in]. This is a contradiction, since [fill in]."

VIII. (5) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

IX. (3) State the Pigeonhole Principle (just the basic Pigeonhole Principle, not the Generalized Pigeonhole Principle).

X. (4) Fill in the missing parts of the following argument, which shows that if one selects any five different numbers between 2 and 9, then some pair of them adds up to 11: "Let $S = \{2,3,4,5,6,7,8,9\}$, and select any five numbers from S. Define $f\colon S \to$ [fill in] by the rule $f(x) = x$ if $2 \leq x \leq 5$ and $f(x) = 11 - x$ if $6 \leq x \leq 9$. By the Pigeonhole Principle, there must be two of the five numbers, call them m and n with $m < n$, for which [fill in]. We cannot have both $m \leq 5$ and $n \leq 5$, since then we would have $m = f(m) = f(n) = n$. Nor can we have $m \geq 6$ and $n \geq 6$, since then we would have [fill in]. So we must have $m \leq 5$ and $n \geq 6$. But then, [fill in], so $m + n = 11$."

XI. (14) This problem concerns strings of 10 digits $d_1 d_2 \cdots d_{10}$, where each $d_i \in \{0, 1, 2, \ldots, 9\}$. Leave the answers to the following questions as expressions which may contain factorials and/or binomial coefficients $\binom{m}{n}$, rather than calculating them out.

1. How many strings of 10 digits are there?
2. How many strings of 10 digits contain no 7?
3. How many strings of 10 digits start with three distinct digits?
4. How many strings of 10 digits have no two equal digits (that is, have all their digits different)?
5. How many strings of 10 digits contain exactly two 4's?
6. How many strings of 10 digits contain exactly two 4's and two 3's?
7. How many strings of 10 digits either start with three 5's or end with two 5's, or both?

XII. (5) Draw Pascal's Triangle down to the row that contains the binomial coefficients $\binom{7}{k}$. Use this row and the Binomial Theorem to write out $(a-1)^7$ (giving the explicit numerical values of the coefficients).

XIII. (7) This problem concerns the identity $n\binom{n-1}{k-1} = k\binom{n}{k}$.

1. Verify the identity by calculation using the formula for $\binom{n-1}{k-1}$.
2. Verify the identity by counting using two different methods the number of ways to choose a subset with k elements from a set of n elements, then choose one of the elements of this subset. That is, let S be a set of n elements and count the number of pairs (A, x) where A has k elements, $A \subseteq S$, and $x \in A$.

XIV. (3) List all the ordered pairs in the relation on the set $A = \{1, 2, 3, 4\}$ defined by $a \mathrel{\mathrm{R}} b \Leftrightarrow a|b$.

XV. (4) Verify that the relation of congruence modulo m is an equivalence relation.

XVI. (6) List explicitly the elements in the congruence classes [0], [1], [2], [3], [4], [5] for the equivalence relation of congruence modulo 6 on the set of integers (show at least five integers from each class). Write out the addition and multiplication tables for the set $\{[0], [1], [2], [3], [4], [5]\}$. Which elements have inverses for multiplication?