Math 2513 homework

26. (10/24) Prove that if there exists d so that $cd \equiv 1 \mod m$, then $\gcd(c, m) = 1$. Hint: use the theorem that says $\gcd(a, b)$ is the least positive sum of multiples of a and b.

27. (10/24) Prove that if $\gcd(c, m) = 1$, then there exists d so that $cd \equiv 1 \mod m$. Hint: use the theorem that says $\gcd(a, b)$ is the least positive sum of multiples of a and b.

28. (10/24) Use the previous problem to prove that if $ac \equiv bc \mod m$ and $\gcd(c, m) = 1$, then $a \equiv b \mod m$.

29. (11/2) Let $m = sa + tb$ be the smallest positive sum of multiples of a and b. Rewrite the argument that we used in class to show that $m|a$ to obtain an argument that shows that $m|b$.

30. (11/2) Use the Euclidean algorithm to verify the following:
    1. For all $n > 0$, $\gcd(n, n + 1) = 1$.
    2. $\gcd(n, n + 2)$ is 1 if n is odd and is 2 if n is even.

31. (11/2) Use the fact that $\gcd(a, b)$ is the smallest positive integer that is a sum of multiples of a and b to verify the following:
    1. For all $n > 0$, $\gcd(n, n + 1) = 1$.
    2. $\gcd(n, n + 2)$ is 1 if n is odd (write $n = 2k + 1$ and show that 1 is a sum of multiples of $2k + 1$ and $2k + 3$) and is 2 if n is even.

32. (11/2) 3.2 # 15, 17, 18, 23, 24, 27, 28-30

33. (11/2) Let $f : A \to B$ and $g : B \to C$ be functions.
    1. Prove that if f and g are injective, then their composition $g \circ f : A \to C$ is also injective.
    2. Prove that if f and g are surjective, then their composition $g \circ f : A \to C$ is also surjective.
    3. Use the previous two facts to prove that if f and g are bijective, then their composition $g \circ f : A \to C$ is also bijective.

34. (11/2) 3.2 # 31, 36

35. (11/2) 3.2 # 38 (arrange the pairs (m, n) analogously to the way we arranged the positive fractions when proving that $\mathbb{Q}$ is countable)

36. (11/2) 3.3 # 20-21 (for each, give a proof using induction and a proof using congruence), 23 (let $P(n)$ be $8|(2n - 1)^2 - 1$)