Math 2513 homework

26. (10/24) Prove that if there exists \(d\) so that \(cd \equiv 1 \mod m\), then \(\gcd(c, m) = 1\). Hint: use the theorem that says \(\gcd(a, b)\) is the least positive sum of multiples of \(a\) and \(b\).

27. (10/24) Prove that if \(\gcd(c, m) = 1\), then there exists \(d\) so that \(cd \equiv 1 \mod m\). Hint: use the theorem that says \(\gcd(a, b)\) is the least positive sum of multiples of \(a\) and \(b\).

28. (10/24) Use the previous problem to prove that if \(ac \equiv bc \mod m\) and \(\gcd(c, m) = 1\), then \(a \equiv b \mod m\).

29. (11/2) Let \(m = sa + tb\) be the smallest positive sum of multiples of \(a\) and \(b\). Rewrite the argument that we used in class to show that \(m|a\) to obtain an argument that shows that \(m|b\).

30. (11/2) Use the Euclidean algorithm to verify the following:
   1. For all \(n > 0\), \(\gcd(n, n + 1) = 1\).
   2. \(\gcd(n, n + 2)\) is 1 if \(n\) is odd and is 2 if \(n\) is even.

31. (11/2) Use the fact that \(\gcd(a, b)\) is the smallest positive integer that is a sum of multiples of \(a\) and \(b\) to verify the following:
   1. For all \(n > 0\), \(\gcd(n, n + 1) = 1\).
   2. \(\gcd(n, n + 2)\) is 1 if \(n\) is odd (write \(n = 2k + 1\) and show that 1 is a sum of multiples of \(2k + 1\) and \(2k + 3\)) and is 2 if \(n\) is even.

32. (11/2) 3.2 # 15, 17, 18, 23, 24, 27, 28-30

33. (11/2) Let \(f: A \to B\) and \(g: B \to C\) be functions.
   1. Prove that if \(f\) and \(g\) are injective, then their composition \(g \circ f: A \to C\) is also injective.
   2. Prove that if \(f\) and \(g\) are surjective, then their composition \(g \circ f: A \to C\) is also surjective.
   3. Use the previous two facts to prove that if \(f\) and \(g\) are bijective, then their composition \(g \circ f: A \to C\) is also bijective.

34. (11/2) 3.2 # 31, 36

35. (11/2) 3.2 # 38 (arrange the pairs \((m, n)\) analogously to the way we arranged the positive fractions when proving that \(\mathbb{Q}\) is countable)

36. (11/2) 3.3 # 20-21 (for each, give a proof using induction and a proof using congruence), 23 (let \(P(n)\) be \(8|(2n - 1)^2 - 1\))