I. Find the Taylor series for the function $f(x) = x^3$ at $a = -1$.

\[ f(-1) = -1, \quad f'(-1) = 3(-1)^2 = 3, \quad f''(-1) = 6(-1) = -6, \quad f'''(-1) = 6, \quad \text{and all higher derivatives are 0}. \] So the Taylor series is $f(-1) + f'(-1)(x+1) + (f''(-1)/2!) (x+1)^2 + (f'''(-1)/3!) (x+1)^3 = -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$.

II. Use power series to calculate each of the following:

1. \( \lim_{x \to 0} \frac{\ln(1 + 3x) - 3x}{x^2} \)

\[
\lim_{x \to 0} \frac{\ln(1 + 3x) - 3x}{x^2} = \lim_{x \to 0} \frac{(3x - (3x)^2/2 + (3x)^3/3 + \cdots) - 3x}{x^2} = \lim_{x \to 0} \frac{-9x^2/2 + 27x^3/3 + \cdots}{x^2} = \lim_{x \to 0} \frac{-9}{2} + 27x/3 + \cdots = -9/2
\]

2. \( \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} \)

\[
\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/6)^{2n}}{(2n)!} = \cos(\pi/6) = \sqrt{3}/2.
\]

3. A numerical series whose value is \( \int_{0}^{1} \frac{\sin(x)}{x} \, dx \) (but do not try to calculate the numerical value of the series).

\[
\int_{0}^{1} \frac{\sin(x)}{x} \, dx = \int_{0}^{1} \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x} \, dx = \int_{0}^{1} 1 - (x^2/3!) + (x^4/5!) - \cdots \, dx = x - x^3/(3!3) + x^5/(5!5) - \cdots \bigg|_{0}^{1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)}
\]

III. For a certain function $f(x)$, the Taylor polynomial $T_5$ of degree 5 at $a = 0$ is $1 + 20x^3 + x^5$. Suppose that one uses the value $T_5(0.1) = 1.02001$ as an approximation for $f(0.1)$. Suppose that it is known that all derivatives of $f$ have values between 0 and 200 at $x$-values in the range $-10 \leq x \leq 10$. Use Taylor’s form $R_n(x) = \int_{a}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t) \, dt$ for the remainder to calculate an upper bound for the error of this approximation (that is, make an upper estimate of $|R_5(0.1)|$).

\[
|R_5(0.1)| = \left| \int_{0}^{0.1} \frac{(0.1-t)^5}{5!} f^6(t) \, dt \right| \leq \left| \int_{0}^{0.1} \frac{200 (0.1-t)^5}{5!} \, dt \right| = \left| \frac{200}{6!} \cdot 0.1^6 \bigg|_{0}^{0.1} \right| = \frac{200}{6!} \cdot 0.1^6 = \frac{200}{6! \cdot 1,000,000} = \frac{1}{6! \cdot 5,000}
\]
We know that $\vec{\imath} \times \vec{\jmath} = -\vec{k}$ and $\vec{k} \times \vec{\imath} = \vec{\jmath}$.

1. Use these facts to find a vector $\vec{v}$ such that $\vec{v} \times \vec{\imath} = \vec{\jmath} + 2 \vec{k}$.
   
   (quick solution) We have $\vec{k} \times \vec{\imath} = \vec{\jmath}$. Also, $\vec{\jmath} \times \vec{\imath} = -\vec{k}$ so $-2\vec{\imath} \times \vec{\imath} = -2(-\vec{k}) = 2\vec{k}$, so we can take $\vec{v} = \vec{k} - 2\vec{\jmath}$.
   
   (more methodical) We want $\vec{\jmath} + 2\vec{k} = (a\vec{\imath} + b\vec{\jmath} + c\vec{k}) \times \vec{\imath} = a\vec{\imath} \times \vec{\imath} + b\vec{\jmath} \times \vec{\imath} + c\vec{k} \times \vec{\imath} = -b\vec{k} + c\vec{\jmath}$; forcing $b = -2$ and $c = 1$, and we may take $\vec{v} = -2\vec{\jmath} + \vec{k}$.

2. Find a vector $\vec{w}$ such that $\vec{w} \times (\vec{w} + 2\vec{\imath})$ equals $\vec{\jmath} + 2\vec{k}$.
   
   We want $\vec{\jmath} + 2\vec{k} = \vec{w} \times (\vec{w} + 2\vec{\imath}) = \vec{w} \times \vec{w} + \vec{w} \times 2\vec{\imath} = 2\vec{w} \times \vec{\imath}$, so we can take $2\vec{w} = \vec{v}$, i.e. $\vec{w} = -\vec{\jmath} + \frac{1}{2} \vec{k}$.

A straight line $L$ has direction vector $2\vec{\imath} - \vec{\jmath} + \vec{k}$ and passes through the point $P = (0, 4, -0.5)$.

1. Write an equation for $L$ as a vector-valued function of $t$.
   $$\vec{r}(t) = 2t\vec{\imath} + (4 - t)\vec{\jmath} + (-0.5 + t)\vec{k}.$$  
   
2. Write parametric equations for $L$.
   $$x = 2t, \quad y = 4 - t, \quad z = -0.5 + t.$$  
   
3. Write equations in symmetric form for $L$.
   $$\frac{x}{2} = \frac{(y - 4)}{(-1)} = \frac{(z + 0.5)}{1}.$$  
   
4. Write an equation for the plane through $P$ perpendicular to $L$.
   
   We can take $\vec{n} = 2\vec{\imath} - \vec{\jmath} + \vec{k}$, so an equation is $2x - (y - 4) + (z + 0.5) = 0$, or $2x - y + z + 4.5 = 0$.
   
5. Find a normal vector to the plane that contains $L$ and also contains the origin.
   
   The plane is parallel to $2\vec{\imath} - \vec{\jmath} + \vec{k}$ and also to the position vector of $P$, which is $4\vec{\jmath} - 0.5\vec{k}$, so a normal vector is the cross product $(2\vec{\imath} - \vec{\jmath} + \vec{k}) \times (4\vec{\jmath} - 0.5\vec{k}) = (-7/2)\vec{\imath} + \vec{\jmath} + 8\vec{k}.$
VI. Four points $O, A, B,$ and $C$ in the $xy$-plane are shown in the figure to the right; the angle $AOB$ is a right angle. Let $\vec{a}, \vec{b},$ and $\vec{c}$ be the vectors from $O$ to $A, B,$ and $C$ respectively.

1. Tell why $\vec{a} \cdot \vec{b} > \vec{a} \cdot \vec{c}$.
   
   Since $\vec{a}$ and $\vec{b}$ are perpendicular, $\vec{a} \cdot \vec{b} = 0$. Since the angle between $\vec{a}$ and $\vec{c}$ is more than a right angle, $\vec{a} \cdot \vec{c}$ is negative.

2. Tell why $\|\vec{a} \times \vec{b}\| < \|\vec{a} \times \vec{c}\|$.

   The area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ is smaller than that of the parallelogram spanned by $\vec{a}$ and $\vec{c}$, since they both have base of length $\|\vec{a}\|$, but the second one has larger altitude.

3. Tell why $(\vec{c} \times \vec{a}) \times \vec{b}$ points in the direction of $\vec{a}$.

   By the right-hand rule, $\vec{c} \times \vec{a}$ is perpendicular to the page, pointing toward the reader. The cross-product of such a vector with $\vec{b}$ is lies again in the $xy$-plane, and is perpendicular to $\vec{b}$; again using the right-hand rule, it points in the direction of $\vec{a}$, rather than in the opposite direction.

4. Tell the approximate direction of $\vec{c} \times (\vec{a} \times \vec{b})$.

   $\vec{a} \times \vec{b}$ is perpendicular to the $xy$-plane, pointing directly away from the reader. So $\vec{c} \times (\vec{a} \times \vec{b})$ lies in the $xy$-plane and is perpendicular to $\vec{c}$, in fact the right-hand rule shows it points approximately in the direction of $\vec{b}$.