Mathematics 2433-001H Name (please print) ____________________

Examination III

November 18, 2005

I. (5) Find the Taylor series for the function $f(x) = x^3$ at $a = -1$.

$f(-1) = -1$, $f'(-1) = 3(-1)^2 = 3$, $f''(-1) = 6(-1) = -6$, $f'''(-1) = 6$, and all higher derivatives are 0. So the Taylor series is $f(-1) + f'(-1)(x+1) + (f''(-1)/2!)(x+1)^2 + (f'''(-1)/3!)(x+1)^3 = -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$.

II. (15) Use power series to calculate each of the following:

1. $\displaystyle\lim_{x\to 0} \frac{\ln(1+3x) - 3x}{x^2}$.

$$\lim_{x\to 0} \frac{\ln(1+3x) - 3x}{x^2} = \lim_{x\to 0} \frac{(3x - (3x)^2/2 + (3x)^3/3 + \cdots) - 3x}{x^2}$$
$$= \lim_{x\to 0} \frac{-9x^2/2 + 27x^3/3 + \cdots}{x^2} = \lim_{x\to 0} -9/2 + 27x/3 + \cdots = -9/2$$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\pi/6)^{2n} = \cos(\pi/6) = \sqrt{3}/2.$$

3. A numerical series whose value is $\displaystyle\int_0^1 \frac{\sin(x)}{x}\,dx$ (but do not try to calculate the numerical value of the series).

$$\int_0^1 \frac{\sin(x)}{x}\,dx = \int_0^1 \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x}\,dx$$
$$= \int_0^1 1 - (x^2/3!) + (x^4/5!) - \cdots\,dx = x - x^3/(3!\,3) + x^5/(5!\,5) - \cdots \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(2n+1)}$$

III. (6) For a certain function $f(x)$, the Taylor polynomial T_5 of degree 5 at $a = 0$ is $1 + 20x^3 + x^5$. Suppose that one uses the value $T_5(0.1) = 1.02001$ as an approximation for $f(0.1)$. Suppose that it is known that all derivatives of f have values between 0 and 200 at x-values in the range $-10 \le x \le 10$. Use Taylor's form $R_n(x) = \displaystyle\int_a^x \frac{(x-t)^n}{n!}\, f^{(n+1)}(t)\,dt$ for the remainder to calculate an upper bound for the error of this approximation (that is, make an upper estimate of $|R_5(0.1)|$).

$$|R_5(0.1)| = \left| \int_0^{0.1} \frac{(0.1-t)^5}{5!}\, f^6(t)\,dt \right| \le \left| \int_0^{0.1} \frac{200\,(0.1-t)^5}{5!}\,dt \right|$$
$$= \left| -\frac{200\,(0.1-t)^6}{6!}\Big|_0^{0.1} \right| = \frac{200\,(0.1)^6}{6!} = \frac{200}{6! \cdot 1{,}000{,}000} = \frac{1}{6! \cdot 5{,}000}$$

IV. We know that $\vec{\jmath} \times \vec{\imath} = -\vec{k}$ and $\vec{k} \times \vec{\imath} = \vec{\jmath}$.
(6)

1. Use these facts to find a vector $\vec{v}$ such that $\vec{v} \times \vec{\imath} = \vec{\jmath} + 2\,\vec{k}$.

   (quick solution) We have $\vec{k} \times \vec{\imath} = \vec{\jmath}$. Also, $\vec{\jmath} \times \vec{\imath} = -\vec{k}$ so $-2\vec{\jmath} \times \vec{\imath} = -2(-\vec{k}) = 2\vec{k}$, so we can take $\vec{v} = \vec{k} - 2\vec{\jmath}$.

   (more methodical) We want $\vec{\jmath} + 2\vec{k} = (a\vec{\imath} + b\vec{\jmath} + c\vec{k}) \times \vec{\imath} = a\,\vec{\imath} \times \vec{\imath} + b\,\vec{\jmath} \times \vec{\imath} + c\,\vec{k} \times \vec{\imath} = -b\vec{k} + c\vec{\jmath}$, forcing $b = -2$ and $c = 1$, and we may take $\vec{v} = -2\vec{\jmath} + \vec{k}$.

2. Find a vector $\vec{w}$ such that $\vec{w} \times (\vec{w} + 2\vec{\imath})$ equals $\vec{\jmath} + 2\,\vec{k}$.

   We want $\vec{\jmath} + 2\vec{k} = \vec{w} \times (\vec{w} + 2\vec{\imath}) = \vec{w} \times \vec{w} + \vec{w} \times 2\vec{\imath} = 2\vec{w} \times \vec{\imath}$, so we can take $2\vec{w} = \vec{v}$, i. e. $\vec{w} = -\vec{\jmath} + \frac{1}{2}\vec{k}$.

V. A straight line L has direction vector $2\vec{\imath} - \vec{\jmath} + \vec{k}$ and passes through the point $P = (0, 4, -0.5)$.
(12)

1. Write an equation for L as a vector-valued function of t.

   $\vec{r}(t) = 2t\,\vec{\imath} + (4 - t)\,\vec{\jmath} + (-0.5 + t)\,\vec{k}$.

2. Write parametric equations for L.

   $x = 2t$, $y = 4 - t$, $z = -0.5 + t$.

3. Write equations in symmetric form for L.

   $x/2 = (y - 4)/(-1) = (z + 0.5)/1$.

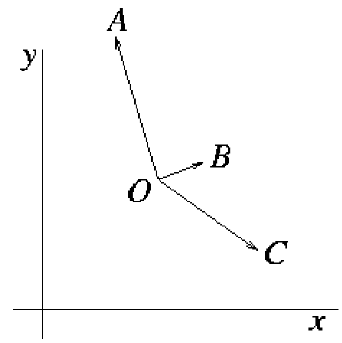
4. Write an equation for the plane through P perpendicular to L.

   We can take $\vec{n} = 2\vec{\imath} - \vec{\jmath} + \vec{k}$, so an equation is $2x - (y - 4) + (z + 0.5) = 0$, or $2x - y + z + 4.5 = 0$.

5. Find a normal vector to the plane that contains L and also contains the origin.

   The plane is parallel to $2\vec{\imath} - \vec{\jmath} + \vec{k}$ and also to the position vector of P, which is $4\vec{\jmath} - 0.5\vec{k}$, so a normal vector is the cross product $(2\vec{\imath} - \vec{\jmath} + \vec{k}) \times (4\vec{\jmath} - 0.5\vec{k}) = (-7/2)\,\vec{\imath} + \vec{\jmath} + 8\,\vec{k}$.

VI. (12) Four points O, A, B, and C in the xy-plane are shown in the figure to the right; the angle AOB is a right angle. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be the vectors from O to A, B, and C respectively.



1. Tell why $\vec{a} \cdot \vec{b} > \vec{a} \cdot \vec{c}$.

   Since $\vec{a}$ and $\vec{b}$ are perpendicular, $\vec{a} \cdot \vec{b} = 0$. Since the angle between $\vec{a}$ and $\vec{c}$ is more than a right angle, $\vec{a} \cdot \vec{c}$ is negative.

2. Tell why $\| \vec{a} \times \vec{b} \| < \| \vec{a} \times \vec{c} \|$.

   The area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ is smaller than that of the parallelogram spanned by $\vec{a}$ and $\vec{c}$, since they both have base of length $\| \vec{a} \|$, but the second one has larger altitude.

3. Tell why $(\vec{c} \times \vec{a}) \times \vec{b}$ points in the direction of $\vec{a}$.

   By the right-hand rule, $\vec{c} \times \vec{a}$ is perpendicular to the page, pointing toward the reader. The cross-product of such a vector with $\vec{b}$ is lies again in the xy-plane, and is perpendicular to $\vec{b}$; again using the right-hand rule, it points in the direction of $\vec{a}$, rather than in the opposite direction.

4. Tell the approximate direction of $\vec{c} \times (\vec{a} \times \vec{b})$.

   $\vec{a} \times \vec{b}$ is perpendicular to the xy-plane, pointing directly away from the reader. So $\vec{c} \times (\vec{a} \times \vec{b})$ lies in the xy-plane and is perpendicular to $\vec{c}$, in fact the right-hand rule shows it points approximately in the direction of $\vec{b}$.