Mathematics 2433-001H Name (please print) ____________________

Examination II

October 24, 2005

I. (20) For each of the following series, use standard facts and/or convergence tests to determine whether the series converges or diverges. Give only brief details, but indicate clearly what fact or test you are using, and give at least the key steps in verifying that the test applies.

(i) $\sum \dfrac{5^{n+2}}{3^{2n}}$

$\sum \dfrac{5^{n+2}}{3^{2n}} = \sum 25\left(\dfrac{5}{9}\right)^n$, and this converges since it is a geometric series with $-1 < \frac{5}{9} < 1$ (one can also use the ratio test)

(ii) $\sum \dfrac{(-1)^n}{2^{1/n}}$

Since $\lim 2^{1/n} = 1$, the terms do not limit to 0, and consequently the series diverges.

(iii) $\sum \tan\left(\dfrac{1}{n}\right)$

$\lim \dfrac{\tan(1/n)}{1/n} = \lim \dfrac{\sin(1/n)}{1/n} \cdot \dfrac{1}{\cos(1/n)} = 1$, so by the Limit Comparison Test, this series has the same convergence behavior as the harmonic series $\sum \frac{1}{n}$, which diverges.

(iv) $\sum \dfrac{1}{n + n\cos^2(n)}$

Since $0 < n + n\cos^2(n) \leq 2n$, we have $0 < \dfrac{1}{2n} \leq \dfrac{1}{n + n\cos^2(n)}$. Since $\sum \dfrac{1}{2n}$ diverges, the Comparison Test shows that $\sum \dfrac{1}{n + n\cos^2(n)}$ diverges.

(the Limit Comparison Test does not apply because $\lim \dfrac{\dfrac{1}{n + n\cos^2(n)}}{\dfrac{1}{n}} = \lim \dfrac{1}{1 + \cos^2(n)}$ does not exist)

(v) $\sum \dfrac{n^4 - 7n^3 + 1}{n^7 - n^4 + 13n}$

$\lim \dfrac{\dfrac{n^4 - 7n^3 + 1}{n^7 - n^4 + 13n}}{1/n^3} = \lim \dfrac{n^7 - 7n^6 + n^3}{n^7 - n^4 + 13n} = \lim \dfrac{1 - 7/n + 1/n^4}{1 - 1/n^3 + 13/n^6} = 1$, so by the Limit Comparison Test, this series has the same convergence behavior as the p-series $\sum \dfrac{1}{n^3}$, which converges.

II. (4) A power series of the form $\sum c_n(x-\pi)^n$ converges at $x = 2$ and diverges at $x = 10$. From this information, what can be determined about its radius of convergence R?

The series is centered at $x = \pi$. Since it converges at $x = 2$, we know that $\pi - 2 \leq R$. Since the series diverges at $x = 10$, we know that $R \leq 10 - \pi$, thus $\pi - 2 \leq R \leq 10 - \pi$.

III. (8) Find the radius of convergence and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}(x+3)^n$.

Applying the Ratio Test, we calculate

$$\lim \frac{|\frac{(-2)^{n+1}}{\sqrt{n+1}}(x+3)^{n+1}|}{|\frac{(-2)^n}{\sqrt{n}}(x+3)^n|} = \lim 2\,\frac{\sqrt{n}}{\sqrt{n+1}}|x+3| = 2\,|x+3|\ .$$

Solving $2\,|x+3| < 1$, we find that the series converges absolutely when $-\frac{7}{2} < x < -\frac{5}{2}$, and diverges when $x < -\frac{7}{2}$ or $-\frac{5}{2} < x$. Therefore the radius of convergence is $\frac{1}{2}$. To determine the exact interval of convergence, we must examine the endpoints $x = -\frac{7}{2}$ and $x = -\frac{5}{2}$.

When $x = -\frac{7}{2}$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges since it is a p-series with $p < 1$.

When $x = -\frac{5}{2}$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Since this series is alternating and the sequence $\dfrac{1}{\sqrt{n}}$ is decreasing and limits to 0, the Alternating Series Test shows that it converges. So the interval of convergence is $-\dfrac{7}{2} < x \leq -\dfrac{5}{2}$.

IV. (6) Fill in the missing parts of the following argument: Suppose that $\sum |a_n|$ converges. Since $0 \leq a_n + |a_n| \leq 2|a_n|$, the Comparison Test shows that [fill in]. Since $\sum (a_n + |a_n|)$ and $\sum$ [fill in] converge, it follows that $\sum$ [fill in] converges.

The argument establishes that absolutely convergent series convege. The missing parts are:

1. $\sum a_n + |a_n|$ converges
2. $-|a_n|$ ($|a_n|$ is acceptable)
3. a_n

V. (5) Let $\sum a_n$ be a series with all $a_n > 0$, and assume that $\lim a_n = 0$. Use the Limit Comparison Test to show that $\sum a_n$ converges if and only if $\sum \ln(1 + a_n)$ converges.

Using l'Hôpital's rule, we calculate that $\displaystyle\lim \frac{\ln(1+a_n)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$, so the Limit Comparison Test shows that the $\sum \ln(1+a_n)$ has the same convergence behavior as $\sum a_n$.

VI. (9) Let $\{a_n\}$ be a sequence. The infinite product $\prod_{n=1}^{\infty} a_n$ is (not surprisingly) defined to be $\lim_{n\to\infty} p_n$ where p_n is the *partial product* defined by $p_n = \prod_{i=1}^{n} a_i$.

1. Calculate $\displaystyle\prod_{n=1}^{\infty} \frac{n}{n+1}$.

   We have $p_n = \frac{1}{2}\,\frac{2}{3}\,\frac{3}{4}\cdots\frac{n}{n+1} = \frac{1}{n+1}$, so $\displaystyle\prod_{n=1}^{\infty} \frac{n}{n+1} = \lim \frac{1}{n+1} = 0$.

2. Show that $\displaystyle\prod_{n=1}^{\infty} 2^{\frac{1}{n^\alpha}}$ converges if and only if $\alpha > 1$.

   We have
   $$p_n = 2^{\frac{1}{1}}\, 2^{\frac{1}{2^\alpha}}\, 2^{\frac{1}{3^\alpha}} \cdots 2^{\frac{1}{n^\alpha}} = 2^{\sum_{k=1}^{n} \frac{1}{k^\alpha}} .$$
   Since $\displaystyle\prod_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges exactly when $\alpha > 1$, its partial sums $\sum_{k=1}^{n} \frac{1}{k^\alpha}$ have a finite limit exactly when $\alpha > 1$. So the partial products p_n, and consequently the infinite product, converge exactly when $\alpha > 1$.

3. Assuming that all $a_n > 0$ and $\lim a_n = 0$, show that $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum a_n$ converges.

   We have $p_n = \displaystyle\prod_{k=1}^{n}(1+a_k)$, so $\ln(p_n) = \sum_{k=1}^{n} \ln(1+a_k)$. These are the partial sums of the series $\sum_{n=1}^{\infty} \ln(1+a_n)$. By problem V above, this series converges exactly when the series $\sum a_n$ converges. So its partial sums $\ln(p_n)$, hence also the partial products p_n, converge exactly when $\sum a_n$ converges.