Mathematics 2433-001H Name (please print) ______________________

Examination I

September 21, 2005

I. (6) The sequence whose n^{th} term is $\dfrac{(-1)^n n^2}{1+n^3}$ converges to 0. State the Squeeze Principle for limits of sequences, and use it to verify that this sequence converges to 0.

The Squeeze Principle says that if $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences whose terms satisfy $a_n \leq b_n \leq c_n$, and $\lim a_n = L$ and $\lim c_n = L$, then $\lim b_n$ exists and equals L.

We have the estimate

$$\frac{n^2}{1+n^3} \leq \frac{n^2}{n^3} = \frac{1}{n} ,$$

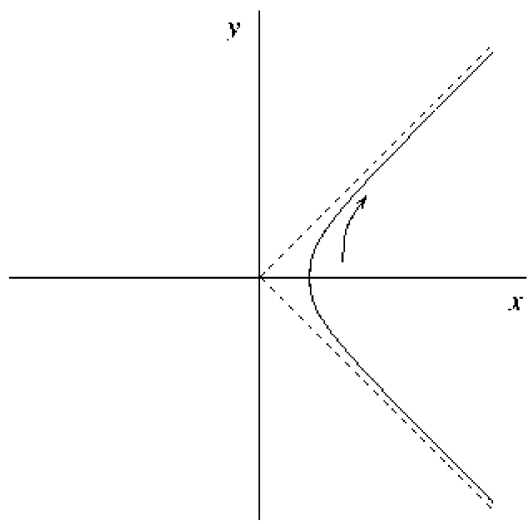
so

$$-\frac{1}{n} \leq \frac{(-1)^n n^2}{1+n^3} \leq \frac{1}{n} .$$

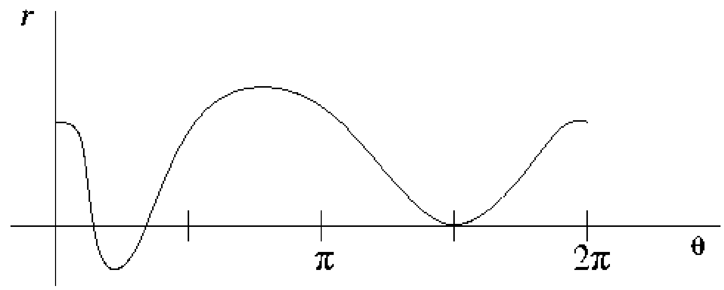
Since $\lim -\dfrac{1}{n}$ and $\lim \dfrac{1}{n}$ are both 0, the Squeeze Principle shows that $\lim \dfrac{(-1)^n n^2}{1+n^3}$ is 0 as well.

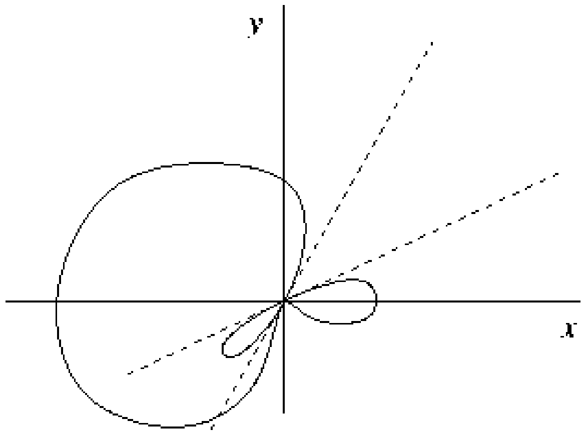
II. (5) Sketch the following curve and indicate with an arrow the direction in which the curve is traced as the parameter moves: $x = \cosh(t)$, $y = \sinh(t)$. Give a brief explanation of why this is the graph.

Since $\cosh^2(t) - \sinh^2(t) = 1$, the points on this curve lie on the hyperbola $x^2 - y^2 = 1$. Since $x = \cosh(t) > 0$, they all lie to the right of the x-axis. So the graph is as shown in the figure below. Also, as t increases, $y = \sinh(t)$ increases, so the direction of motion is as shown.

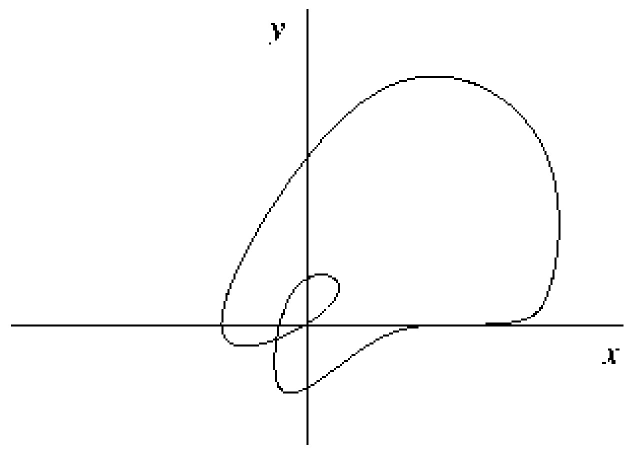


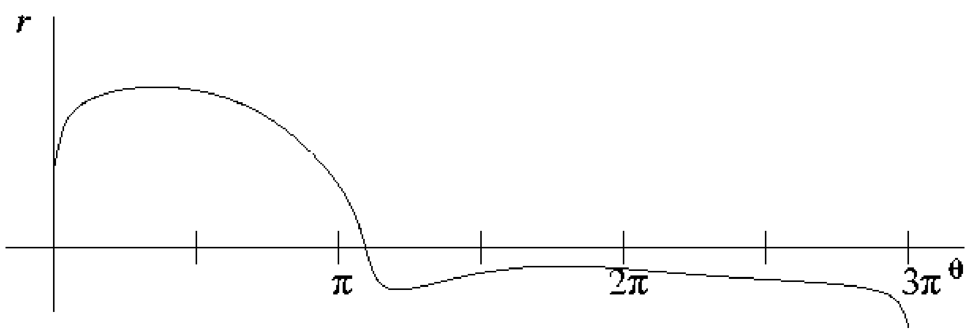
III. (5) The graph of a certain equation $r = f(\theta)$ is shown at the right, in a rectangular θ-r coordinate system. In an x-y coordinate system, make a reasonably accurate graph of the polar equation $r = f(\theta)$ for this function.





IV. (5) The graph of a certain polar equation $r = f(\theta)$ is shown at the right, in an rectangular x-y coordinate system. In a rectangular θ-r coordinate system, make a reasonably accurate graph of the rectangular equation $r = f(\theta)$ for this function. Assume that $r = 1$ when $\theta = 0$.





V. Give (without extensive verification) examples of the following:
(6)

(a) A sequence a_n such that $\{|a_n|\}$ is monotonic, but $\{a_n\}$ is not monotonic.

$\left\{\left(-\frac{2}{3}\right)^n\right\}$, whose sequence of absolute values $\left\{\left(\frac{2}{3}\right)^n\right\}$ is decreasing.

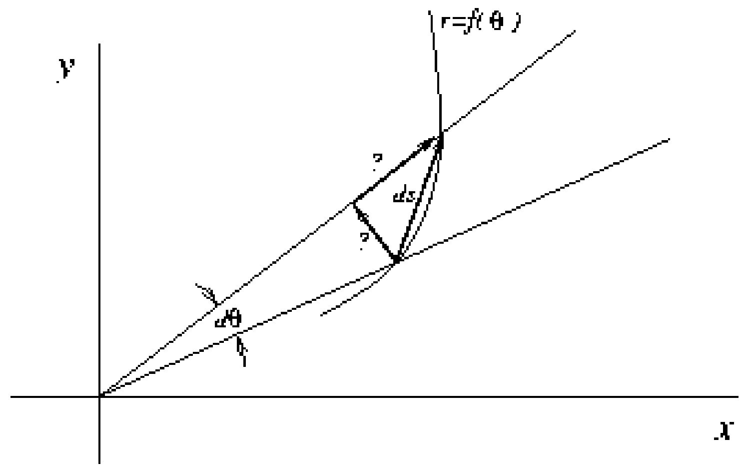
(b) A sequence a_n such that $\{a_n\}$ is monotonic, but $\{|a_n|\}$ is not monotonic.

$\left\{-1, 0, 1, 2, 3, 4, \ldots\right\}$, an increasing sequence whose sequence of absolute values $\left\{1, 0, 1, 2, 3, 4, \ldots\right\}$ is not monotonic.

VI. A certain decreasing sequence $\{a_n\}$ has all $a_n > 0$. Explain how one knows that it must converge.
(4)

Since all terms are positive, the sequence is bounded below by $M = 0$. Since it is also decreasing, the Monotonic Sequence Theorem guarantees that it converges.

VII. The figure to the right shows the graph of a polar equation $r = f(\theta)$ in the x-y plane, and an arrow representing the differential of arclength ds, express the two differentials indicated by ? in terms of the differential $d\theta$, and then use them and the Pythagorean theorem to calculate ds in terms of $d\theta$.
(7)



The upper question mark is the change in r corresponding to $d\theta$, that is, $\dfrac{dr}{d\theta}\,d\theta$. The lower one is the arc subtended at radius r by an angle $d\theta$, that is, $r\,d\theta$. Using the Pythagorean Theorem, we find that

$$ds^2 = \left(\frac{dr}{d\theta}\,d\theta\right)^2 + (r\,d\theta)^2 = \left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)d\theta^2 \ ,$$

so $ds = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\ d\theta$.

VIII. The line $y = x$ is parameterized by $x = t^3/3$, $y = t^3/3$ for t in the domain of all real numbers.
(8)

(a) Calculate ds and use it to find the distance traveled between times $t = -1$ and $t = 1$.

$ds^2 = dx^2 + dy^2 = (t^2\,dt)^2 + (t^2\,dt)^2 = 2t^4\,dt^2$, so $ds = \sqrt{2}|t^2|dt = \sqrt{2}t^2dt$, using the fact that $t^2 \geq 0$. Integrating ds, we find the distance traveled to be $\int_{-1}^{1} \sqrt{2}t^2\,dt = \frac{2\sqrt{2}}{3}$.

(b) The chain rule $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ gives the expression $\frac{dy}{dx} = \frac{t^2}{t^2}$. This expression is undefined when $t = 0$. Using the interpretation of the parametric equations as describing the motion of a point P that moves with coordinates $(x, y) = (t^3/3, t^3/3)$, explain why it is reasonable for $\frac{dy}{dx}$ to be undefined when $t = 0$.

At the moment $t = 0$, the point is momentarily stopped (since its velocity $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{2}\,t^2$ is 0 when $t = 0$. But $\frac{dy}{dx}$ in terms of t is the ratio of the vertical velocity to the horizontal velocity, which measures the *direction* of motion. Since P is stopped at $t = 0$, it does not really have a well-defined direction of motion at that instant.

IX. A point P moves according to the *polar* parametric equations $\theta = \sin(t)$, $r = t$. Describe the motion for $0 \leq t \leq 314.159$. A sketch will certainly be helpful, as will the fact that 1 radian is approximately 57 degrees.
(5)

Since $r = t$, the distance from the origin increases uniformly. But the polar angle $\theta = \sin(t)$ oscillates between 1 radian and -1 radian, approximately 50 times. This gives a graph like the one shown here: